

Seesaw neutrino masses with a second Higgs doublet added

D. Jurčiukonis,^a T. Gajdosik,^{a,b} and A. Juodagalvis^a

^a *Vilnius University, Institute of Theoretical Physics and Astronomy,
A. Goštauto st. 12, Vilnius 01108, Lithuania*

^b *Vilnius University, Physics Faculty,
Saulėtekio al. 9, Vilnius 10222, Lithuania*

E-mail: darius.jurciukonis@tfai.vu.lt

ABSTRACT: We study parameters of an extended standard model. The neutrino sector is enlarged by one or two right-handed singlet fields and the Higgs sector contains one additional doublet. One-loop radiative corrections generate the mass for the light neutrino fields. The numerical analysis is performed varying the masses of heavy neutrinos and of the additional neutral Higgses. The parameters of the neutrino sector, allowing for the seesaw type-I mechanism, are restricted by experimental neutrino oscillation data. Both normal and inverted hierarchies of the light neutrino masses are discussed.

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1 Introduction

The precise interpretation of the neutral lepton fields in the particle physics Lagrangian is not settled yet, owing to the very small mass of the known neutrinos and the weakness of their interaction with other particles [1]. The observed neutrino oscillations support the notion that neutrinos have non-vanishing masses, calling for a modification of the Standard Model (SM). The size of the neutrino mass is not the only puzzle to solve. Absence of an electrical charge allows neutrinos to be their own antiparticles. The nature of the neutrinos – whether they are Dirac or Majorana particles – might be determined by future experiments.

The Standard Model considers neutrinos as massless. Adding heavy right-handed neutral singlets and additional Higgs doublets, the authors of ref. [2] combined the seesaw mechanism (type-I) with the radiative mass generation. The spontaneous symmetry breaking (SSB) of the SM gauge group leads to a Dirac mass term for neutrinos. The assumption that neutrinos are Majorana particles allows an additional term in the Lagrangian, namely, the Majorana mass term for the heavy singlets.

The model parameters allow small masses of the light neutrinos that are compatible with the experimental observations. We use this model in the formulation of Grimus and Lavoura [3, 4], restricting the number of additional Higgs doublets to one. We consider only 1-loop corrections to the neutrino mass matrix. The case of three additional heavy neutrinos added to three light neutrinos was studied e.g. in ref. [5, 6]. We assume either one or two heavy neutrinos. Our preliminary results were presented at several conferences [7–11]. Here we provide a more complete description of the performed numerical analysis.

Our extended model has several subsets of the parameters. The neutrino sector is characterized by the masses of the heavy neutrinos (either one or two), and the strength of the coupling to the neutral Higgs fields. The masses of three light neutrinos are the result of our model parameters. They are subject to experimental constraints, namely the experimental neutrino mass differences, Δm_{\odot}^2 and Δm_{atm}^2 , as well as the experimental neutrino oscillation angles θ_{12} , θ_{13} , and θ_{23} [12]. To estimate the neutrino oscillation angles from the neutrino mixing matrix we follow the ideas of ref. [13]. More details are given in appendix B. It should be noted that experimental data is usually interpreted in the “ 3×3 ” neutrino mixing model [1, 12], i.e. three flavoured neutrinos are considered as mixed states of three neutrino mass-eigenstates. We did not attempt to reinterpret the results in the context of an extended neutrino model. We expect the effect to be negligible.

We parametrize the Higgs sector along the analysis of [14]. The Yukawa couplings are parametrized similarly to Grimus and Lavoura [3, 4], which coincide with [14] in the Higgs sector. For the numerical analysis we take the mass of the SM Higgs boson as $m_{H^0} = 125$ GeV [15] and allow the masses of two heavier Higgs bosons to vary in the range from 126 to 3000 GeV.

The outline of the paper is following. Section 2 reviews the seesaw mechanism and the formalism of the two-Higgs-doublet model as used in our analysis. Sections 3 and 4 describe our main results, namely, the calculated light neutrino mass spectra and the analysis of free parameters. Our findings are summarized in section 5. For completeness, the appendix section A describes the features of the weight vectors b_i that relate the scalar Higgs fields to their mass eigenfields, and section B gives the details of the oscillation angle calculation.

2 Description of the model

We discuss an extension of the Standard Model with enlarged Higgs and neutrino sectors. Our main interest is the neutrino sector. Since we need the Higgs sector for the radiative neutrino masses, we give a short overview of the properties of the Higgs sector that we use in our calculations.

2.1 The Higgs sector

The authors of ref. [16] discuss the basis independent formulation of the general two-Higgs-doublet model (2HDM). Using their definition of the Higgs basis, we can write the two

complex doublets of our model in a unique way

$$\phi_1 = \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}(v + \mathcal{H}_{1r}^0 + iG^0) \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} \mathcal{H}^+ \\ \frac{1}{\sqrt{2}}(\mathcal{H}_{2r}^0 + i\mathcal{H}_{2i}^0) \end{pmatrix}, \quad (2.1)$$

where the vacuum expectation value (vev) $v \simeq 246$ GeV and the Goldstone bosons G^0 and G^+ appear only in the first Higgs doublet ϕ_1 . The relations between the basis independent parameters defining the Higgs potential and the parameters describing the physical states are linear and can be easily inverted. This feature allows us to use the vev, the masses of the physical Higgs bosons, $m_{H_1^0}$, $m_{H_2^0}$, $m_{H_3^0}$, and m_{H^+} , and their mixing angles α_{12} and α_{13} as input parameters.

The mass eigenstate for the charged Higgs boson corresponds directly to the field \mathcal{H}^+ with the mass m_{H^+} , but the mass eigenstates for the neutral Higgs bosons with the masses $m_{H_1^0}$, $m_{H_2^0}$, and $m_{H_3^0}$, respectively, are linear superpositions of the neutral fields \mathcal{H}_{1r}^0 , \mathcal{H}_{2r}^0 , and \mathcal{H}_{2i}^0 . Following the formulation of Grimus and Lavoura [3, 4] these linear superpositions are conveniently expressed by

$$h_k^0 = \phi_{b_k}^0 = \sqrt{2} \operatorname{Re}(b_k^\dagger \bar{\phi}^0) = \sqrt{2} \sum_{j=1}^{n_H} \operatorname{Re}(b_{kj}^* \bar{\phi}_j^0) = \frac{1}{\sqrt{2}} \sum_{j=1}^{n_H} (b_{kj}^* \bar{\phi}_j^0 + b_{kj} \bar{\phi}_j^{0*}), \quad (2.2)$$

where $\bar{\phi}^0$ are the neutral parts of the Higgs doublets without the vev: $\bar{\phi}_1^0 = \phi_1^0 - v/\sqrt{2}$ and $\bar{\phi}_2^0 = \phi_2^0$. The "b-vectors" are $2n_H$ unit vectors $b_k \in \mathbb{C}^{n_H}$ of dimensions $n_H \times 1$. We discuss those vectors in the general case in appendix A. There we also show how to obtain the following parametric values for the vectors b :

$$b_{G^0} = \begin{pmatrix} i \\ 0 \end{pmatrix}, \quad b_1 = \begin{pmatrix} c_{12}c_{13} \\ -s_{12} - ic_{12}s_{13} \end{pmatrix}, \quad b_2 = \begin{pmatrix} s_{12}c_{13} \\ c_{12} - is_{12}s_{13} \end{pmatrix}, \quad b_3 = \begin{pmatrix} s_{13} \\ ic_{13} \end{pmatrix}, \quad (2.3)$$

where $c_{1j} = \cos \alpha_{1j}$ and $s_{1j} = \sin \alpha_{1j}$ are given by the angles that describe the mixing of the neutral Higgs fields.

Restricting ourselves to CP conserving cases we use the analysis of ref. [14], where the authors discuss the CP-invariant Higgs potential in the 2HDM framework under various basis-independent conditions. The possible overall phase, that can be written in front of the second Higgs doublet and that acts like a mixing angle α_{23} between \mathcal{H}_{2r}^0 and \mathcal{H}_{2i}^0 , is used to define the CP-property of the mass eigenstates, corresponding to their coupling to fermions, taking H_2^0 to be CP-even and H_3^0 to be CP-odd. This justifies the distinction between the fields h_k^0 and H_k^0 .

Having a fixed SM Higgs mass $m_{H_1^0}$ and assuming it to be smaller than the other two, non-degenerate neutral Higgs boson masses, we have four conditions (case I, case II, case IIIa with $m_{H_2^0} < m_{H_3^0}$, and case IIIb with $m_{H_2^0} > m_{H_3^0}$), which are listed in Table 1. A study reported in [16] suggests that generality is not lost assuming $-\frac{\pi}{2} \leq \alpha_{12}, \alpha_{13} < \frac{\pi}{2}$. We performed the numerical analysis of the neutrino mass spectrum considering the named cases. In some situations we refer to those cases as "scenarios."

	I	II	III
	$\alpha_{12} = 0$	$\alpha_{13} = 0$	$\alpha_{12} = 0$ $\alpha_{13} = 0$
	$m_{H_2^0} < m_{H_3^0}$	$m_{H_2^0} > m_{H_3^0}$	(a) $m_{H_2^0} < m_{H_3^0}$ (b) $m_{H_2^0} > m_{H_3^0}$
b_1	$\begin{pmatrix} c_{13} \\ -is_{13} \end{pmatrix}$	$\begin{pmatrix} c_{12} \\ -s_{12} \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
b_2	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} s_{12} \\ c_{12} \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
b_3	$\begin{pmatrix} s_{13} \\ ic_{13} \end{pmatrix}$	$\begin{pmatrix} 0 \\ i \end{pmatrix}$	$\begin{pmatrix} 0 \\ i \end{pmatrix}$

Table 1. Basis-independent conditions for a CP-conserving 2HDM scalar potential and vacuum [14]. α_{ij} label the mixing angles of neutral Higgses, $m_{H_2^0}$ and $m_{H_3^0}$ denote the masses for CP-even and CP-odd Higgses.

2.2 The Yukawa couplings

Using the vector-and-matrix notation, the Yukawa Lagrangian for the leptons is expressed by [3, 4]

$$\mathcal{L}_Y = - \sum_{k=1}^{n_H=2} \left(\phi_k^\dagger \bar{\ell}_R \Gamma_k + \tilde{\phi}_k^\dagger \bar{\nu}_R \Delta_k \right) \begin{pmatrix} \nu_L \\ \ell_L \end{pmatrix} + \text{H.c.}, \quad (2.4)$$

where $\tilde{\phi}_k = i\tau_2 \phi_k^*$. The quantities ℓ_R and ν_R are the vectors of the right-handed charged leptons and the right-handed projection of the neutrino singlets, respectively. ℓ_L and ν_L form the lepton doublet under the weak interactions and combine with the Higgs doublets ϕ_k to form $SU(2)_{\text{weak}}$ -invariant terms. They are also vectors in generation-space, $n_L = 3$ denoting the three generations of the SM. The Yukawa coupling matrices Γ_k have a dimension $n_L \times n_L$, while Δ_k have a dimension $n_R \times n_L$, where n_R is the number of singlet neutrino fields.

Taking the bilinear terms of eq. (2.4), which means taking only the vev from the Higgs doublets, we get the Dirac mass terms for charged leptons and neutrinos:

$$M_\ell = \frac{v}{\sqrt{2}} \Gamma_1 \doteq \text{diag}(m_e, m_\mu, m_\tau) \quad (2.5)$$

and

$$M_D = \frac{v}{\sqrt{2}} \Delta_1. \quad (2.6)$$

These have to be diagonalized using a singular-value decomposition (SVD) like in the SM to get the correct definition for the mass eigenstates that will describe the physical particles. Having done this transformation to the mass eigenstates, which we write down as the fields appearing in eq. (2.4), the respective transformation matrices reappear in two

unique combinations, V_{CKM} and V_{PMNS} , in the interactions with the charged gauge bosons W^\mp or the charged scalar bosons H^+ and G^+ , giving the charged current Lagrangian

$$\mathcal{L}_{\text{cc}} = \frac{g}{\sqrt{2}} W_\mu^- \bar{\ell}_L \gamma^\mu P_L V_{\text{PMNS}} \nu_L + \text{H.c.} , \quad (2.7)$$

where g is the $SU(2)$ gauge coupling constant. We give this part of the Lagrangian only as a reference, to show what neutrino experiments measure, as this PMNS matrix V_{PMNS} is the basis for the interpretation of experimental data in the “ 3×3 ” neutrino mixing model [1].

2.3 Neutrinos at tree level

The singlet neutrinos, added to the SM, are neutral with respect to all gauge groups of the SM. This offers the possibility that they are Majorana particles, allowing to write a Majorana mass term for them. Since the Lagrangian has to be a scalar with respect to Lorentz transformations, we have to combine a spinor with itself in a Lorentz invariant way. The only chance for Dirac spinors is to use the charge conjugation matrix \mathbf{C} , which also appears in the definition of the Lorentz covariant conjugation¹ for spinors

$$\hat{\psi} := \gamma^0 \mathbf{C} \psi^* = -\mathbf{C} \bar{\psi}^\top . \quad (2.8)$$

The Majorana condition can now be written as

$$\hat{\psi}_M = \eta_\psi \psi_M , \quad (2.9)$$

where η_ψ is the Majorana phase. Assuming ν_R to be n_R Majorana fermions we can write down a Majorana mass term as

$$\mathcal{L}_{\text{Majorana-mass}} = -\frac{1}{2} \bar{\nu}_R M_R \hat{\nu}_R + \text{H.c.} = \frac{1}{2} \bar{\nu}_R M_R \mathbf{C} \bar{\nu}_R^\top + \text{H.c.} , \quad (2.10)$$

where the order of M_R and \mathbf{C} is irrelevant, as these matrices act on different indices of the spinor ν_R : \mathbf{C} is a 4×4 -Dirac matrix, connecting the spinor indices of ν_R , whereas M_R is a symmetric $n_R \times n_R$ matrix, acting on the “generation” index of ν_R . Since the mechanism generating the Majorana mass is not known, we assume the singlets ν_R to be already in the mass eigenstate of M_R . This means, we assume M_R to be diagonal, containing the Majorana masses of the heavy singlets: $M_R = \hat{M}_R$.

Together with the Dirac mass, coming from the Yukawa terms eq. (2.4), the mass terms for the neutrinos are

$$\begin{aligned} \mathcal{L}_{\nu\text{-mass}} &= -\bar{\nu}_R M_D \nu_L - \frac{1}{2} \bar{\nu}_R M_R \hat{\nu}_R + \text{H.c.} \\ &= -\frac{1}{2} \bar{\nu}_R M_D \nu_L - \frac{1}{2} \bar{\tilde{\nu}}_L M_D^\top \hat{\nu}_R + \frac{1}{2} \bar{\nu}_R M_R \mathbf{C} \bar{\nu}_R^\top + \text{H.c.} \\ &= -\frac{1}{2} \begin{pmatrix} \bar{\tilde{\nu}}_L & \bar{\nu}_R \end{pmatrix} \begin{pmatrix} 0 & M_D^\top \\ M_D & \hat{M}_R \end{pmatrix} \begin{pmatrix} \nu_L \\ \hat{\nu}_R \end{pmatrix} + \text{H.c.} \end{aligned} \quad (2.11)$$

¹For a very clear and exhaustive description of the difference between Majorana and Dirac spinors, see ref. [17].

and can be written in a compact form by introducing an $(n_L + n_R) \times (n_L + n_R)$ symmetric neutrino mass matrix

$$M_\nu = \begin{pmatrix} 0 & M_D^\top \\ M_D & \hat{M}_R \end{pmatrix}. \quad (2.12)$$

The neutrino mixing matrix M_ν can be diagonalized [2, 4] using the properties of the singular-value decomposition of a symmetric matrix

$$U^\top M_\nu U = \hat{m} = \text{diag} \left(m_{l_1}, m_{l_2}, m_{l_3}, m_{h_1}, \dots, m_{h_{n_R}} \right), \quad (2.13)$$

where m_{l_i} and m_{h_i} are real and non-negative with mass-ordering $m_{l_1} \leq m_{l_2} \leq m_{l_3}$ and $m_{h_1} \leq \dots \leq m_{h_{n_R}}$. In order to implement the seesaw mechanism [18, 19] we assume that the elements of M_D are of order m_D and those of M_R are of order m_R , with $m_D \ll m_R$. Then, the neutrino masses m_{l_i} with $i = 1, \dots, n_L$ (where $n_L = 3$), are of order m_D^2/m_R , while the masses m_{h_i} with $i = 1, n_R$ (where $n_R = 1$ or 2), are of order m_R . It is useful to decompose the $(n_L + n_R) \times (n_L + n_R)$ unitary matrix U as [2, 4]

$$U = \begin{pmatrix} U_L \\ U_R^* \end{pmatrix}, \quad (2.14)$$

where the submatrix U_L is $n_L \times (n_L + n_R)$ and the submatrix U_R is $n_R \times (n_L + n_R)$. These submatrices obey certain unitarity relations:

$$U_L U_L^\dagger = \mathbb{1}_{n_L}, \quad U_R U_R^\dagger = \mathbb{1}_{n_R}, \quad U_L U_R^\top = 0_{n_L \times n_R} \quad \text{and} \quad U_L^\dagger U_L + U_R^\top U_R^* = \mathbb{1}_{n_L + n_R}. \quad (2.15)$$

Combining with eq. (2.13), we can obtain the following relations:

$$U_L^* \hat{m} U_L^\dagger = 0, \quad U_R \hat{m} U_L^\dagger = M_D \quad \text{and} \quad U_R \hat{m} U_R^\top = M_R. \quad (2.16)$$

With these submatrices of U , the left- and right-handed neutrinos can be written as linear superpositions of the $n_L + n_R$ physical Majorana neutrino fields χ_i :

$$\nu_L = U_L P_L \chi \quad \text{and} \quad \hat{\nu}_R = U_R^* P_L \chi \quad \text{or} \quad \nu_R = U_R P_R \chi, \quad (2.17)$$

where P_L and P_R are the projectors of chirality.

Switching to the physical Majorana mass states χ , we have to express the couplings using the matrices U_L and U_R . The loop corrections, described in the next subsection, depend on the neutrino couplings to the Z -boson and to the neutral Higgses. Interaction with the Z boson is given by

$$\mathcal{L}_{\text{nc}}^{(\nu)} = \frac{g}{4c_w} Z_\mu \bar{\chi} \gamma^\mu \left[P_L \left(U_L^\dagger U_L \right) - P_R \left(U_L^\top U_L^* \right) \right] \chi, \quad (2.18)$$

where c_w is the cosine of the Weinberg angle. The Yukawa couplings for the neutral scalars take the form

$$\begin{aligned} \mathcal{L}_Y^{(\nu)} (h_k^0) = & -\frac{1}{2\sqrt{2}} \sum_{k=1}^{2n_H=4} h_k^0 \bar{\chi} \left[\left(U_R^\dagger \Delta_{b_k} U_L + U_L^\top \Delta_{b_k}^\top U_R^* \right) P_L \right. \\ & \left. + \left(U_L^\dagger \Delta_{b_k}^\dagger U_R + U_R^\top \Delta_{b_k}^* U_L^* \right) P_R \right] \chi, \end{aligned} \quad (2.19)$$

where we treat the Goldstone boson G^0 as h_4^0 . The Yukawa coupling Δ_{b_k} is the result of rewriting the Yukawa Lagrangian eq. (2.4) using the physical Higgs fields defined in eq. (2.2):

$$\Delta_{b_k} = \sum_{j=1}^{n_H} (b_k)_j \Delta_j. \quad (2.20)$$

2.4 Loop corrections to the neutrino masses

We are interested in radiatively generated neutrino masses at one-loop level. The largest influence from the corrections to the neutrino mass matrix has the neutrino Majorana mass term δM_L , since this submatrix is zero at tree level, $M_L|_{\text{tree}} = 0$. The contributions from charge-changing currents are subdominant [3, 4].

We calculate the radiative light neutrino masses following ref. [3]. Once the one-loop corrections are taken into account the neutral fermion mass matrix is given by

$$M_\nu^{(1)} = \begin{pmatrix} \delta M_L & M_D^\top + \delta M_D^\top \\ M_D + \delta M_D & \hat{M}_R + \delta M_R \end{pmatrix} \approx \begin{pmatrix} \delta M_L & M_D^\top \\ M_D & \hat{M}_R \end{pmatrix}, \quad (2.21)$$

where the $0_{3 \times 3}$ matrix appearing at tree level (2.12) is replaced by a symmetric matrix δM_L . This correction dominates among all the sub-matrices of corrections. The one-loop corrections to δM_L originate via the self-energy function $\Sigma_L^{S(X)}(0)$ (where $X = Z, G^0, H_b^0$) that arises from the self-energy Feynman diagrams. The contributions $\Sigma_L^S(p^2)$ are evaluated at zero external momentum squared ($p^2 = 0$). The neutrino couplings to the Z , Higgs H_b^0 and Goldstone G^0 bosons are determined by eqs. (2.18) and (2.19). Each diagram contains a divergent piece but the sum of the three contributions yields a finite result. The expression for one-loop corrections is given by (see e.g. [3])

$$\begin{aligned} \delta M_L = & \sum_{k=1}^3 \frac{1}{32\pi^2} \Delta_{b_k}^\top U_R^* \hat{m} \left(\frac{\hat{m}^2}{m_{H_k^0}^2} - \mathbb{1} \right)^{-1} \ln \left(\frac{\hat{m}^2}{m_{H_k^0}^2} \right) U_R^\dagger \Delta_{b_k} \\ & + \frac{3g^2}{64\pi^2 m_W^2} M_D^\top U_R^* \hat{m} \left(\frac{\hat{m}^2}{m_Z^2} - \mathbb{1} \right)^{-1} \ln \left(\frac{\hat{m}^2}{m_Z^2} \right) U_R^\dagger M_D, \end{aligned} \quad (2.22)$$

where the sum index k runs over all neutral physical Higgses H_k^0 .

3 Results for the case $\mathbf{n_R} = 1$

3.1 General parameterization

First we consider a minimal extension of the standard model by adding only one right-handed neutrino field ν_R to three left-handed fields ν_L . This simple model is useful because it has a small number of parameters. It allows us to obtain relations between the free parameters and the light neutrino masses.

For the study using the general parameterization (reported in [7]), we fix the Higgs sector by choosing a specific CP-conserving set of vectors b :

$$b_{G^0} = \begin{pmatrix} i \\ 0 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 \\ i \end{pmatrix}, \quad b_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (3.1)$$

We use a parameterization of the Yukawa matrices Δ_1 and Δ_2 in the following form:

$$\Delta_i = \frac{\sqrt{2}m_D}{v} \vec{a}_i^\top, \quad i = 1, 2. \quad (3.2)$$

We further assume that the linearly independent vectors are normalized, $|\vec{a}_1| = |\vec{a}_2| = 1$, and equal total strength of the couplings. Using the block form, diagonalization of the symmetric neutrino mass matrix at tree level $M_\nu^{(0)}$ (2.12) can be written as

$$U_{\text{tree}}^T M_\nu^{(0)} U_{\text{tree}} = U_{\text{tree}}^T \begin{pmatrix} 0_{3 \times 3} & m_D \vec{a}_1 \\ m_D \vec{a}_1^\top & \hat{M}_R \end{pmatrix} U_{\text{tree}} = \begin{pmatrix} \hat{M}_l^{(0)} & 0 \\ 0 & \hat{M}_h^{(0)} \end{pmatrix}. \quad (3.3)$$

The non-zero masses in $\hat{M}_l^{(0)}$ and $\hat{M}_h^{(0)}$ can be determined analytically by finding the eigenvalues of the hermitian matrix $M_\nu^{(0)} M_\nu^{(0)\dagger}$. These eigenvalues are squares of the neutrino masses, $\hat{M}_l^{(0)} = \text{diag}(0, 0, m_l^{(0)})$ and $\hat{M}_h^{(0)} = m_h^{(0)}$. The solutions

$$m_D^2 = m_h^{(0)} m_l^{(0)}, \quad (3.4)$$

$$m_R^2 = \left(m_h^{(0)} - m_l^{(0)}\right)^2 \approx \left(m_h^{(0)}\right)^2 \quad (3.5)$$

correspond to the seesaw mechanism.

We diagonalize the tree-level neutrino mass matrix $M_\nu^{(0)}$ using a diagonalization matrix U_{tree} made of two diagonal matrices of phases and three rotation matrices:

$$U_{\text{tree}} = \hat{U}_\phi(\phi_i) U_{12}(\alpha_1) U_{23}(\alpha_2) U_{34}(\beta) \hat{U}_i, \quad (3.6)$$

The angle β is determined by the masses $m_l^{(0)}$ and $m_h^{(0)}$: $\tan^2(\beta) = m_l^{(0)}/m_h^{(0)}$. The dependency of ϕ_i and α_i on m_D and \vec{a}_1 may be expressed analytically. The diagonalization matrix U_{tree} is unitary because the rotation matrices U_{ij} and the diagonal phase matrices \hat{U}_ϕ and \hat{U}_i are all unitary.

Diagonalization of the neutrino mass matrix with the one-loop corrections included, $M_\nu^{(1)}$ in eq. (2.21), is performed numerically using a unitary matrix

$$U_{\text{loop}} = U_{\text{egv}} \hat{U}_\varphi(\varphi_1, \varphi_2, \varphi_3), \quad (3.7)$$

where U_{egv} is an eigenmatrix of $M_\nu^{(1)\dagger} M_\nu^{(1)}$, and \hat{U}_φ is a phase absorption matrix. As discussed already in ref. [2], the heaviest light neutrino obtains mass at tree level from the seesaw mechanism. The second light neutrino obtains mass from radiative corrections. The lightest neutrino remains massless.

The numerical evaluation of the model parameters and of the masses of the light neutrinos proceeds in several steps. First, the mass matrix for the tree level is constructed. The lightest neutrino remains massless in this model, $m_{l_1} = 0$. Using the central values of the experimental neutrino mass differences, we estimate the mass of the heaviest light neutrino and use it as an input parameter m_l^{in} . The value of m_R is another input parameter. Using the seesaw relations (3.4) and (3.5), we evaluate $m_D = \sqrt{m_R m_l^{\text{in}}}$. The vector \vec{a}_1 is generated randomly. This fully determines the tree-level neutrino mass matrix $M_\nu^{(0)}$.

Solving the eigenvalue equation (3.3) we obtain the tree-level neutrino masses $m_l^{(0)}$, $m_h^{(0)}$, and the diagonalization matrix U_{tree} . The next step is to evaluate the one-loop corrections to the neutrino mass matrix δM_L . Additional input parameters of \vec{a}_2 enter the procedure. Diagonalization of the corrected neutrino mass matrix $M_\nu^{(1)}$ yields masses for two light neutrinos. If the obtained mass differences are compatible with the experimental data on neutrino oscillations, the model parameter set is kept. Otherwise, another set of input parameters $\{m_R, \vec{a}_1, \vec{a}_2, m_{H_{2,3}^0}\}$ is generated.

The study suggested a lower limit of 830 GeV for the mass of the heavy singlet [7].

3.2 Reduced parameterization

For the Higgs sector we use the values of the orthogonal complex vectors b listed in Table 1. The mass of the lightest neutral Higgs is fixed at $m_{H_1^0} = 125$ GeV. The masses of heavier neutral Higgses $m_{H_2^0}$ and $m_{H_3^0}$ are generated randomly in the range from 126 to 3000 GeV.

The light neutrino fields can be transformed in such a way that $\vec{a}_1^\top = (0, 0, 1)$, and $\vec{a}_2^\top = (0, n, n')$ with real numbers $m_D, n > 0$, and a complex number n' . Due to the assumed normalisation conditions $|\vec{a}_1| = |\vec{a}_2| = 1$, there are only two independent parameters, namely, the real number n ($n \leq 1$), and a complex phase ϕ :

$$\vec{a}_1^\top = (0, 0, 1) , \quad (3.8)$$

$$\vec{a}_2^\top = (0, n, e^{i\phi} \sqrt{1 - n^2}) . \quad (3.9)$$

A similar case was studied in sect. 4 of ref. [2] without the assumption $|\vec{a}_2| = 1$. Since the goal was to demonstrate that one of the massless neutrinos can obtain mass through the 1-loop radiative corrections, only the expressions are given there. In addition, the masses of the lightest and the heaviest light neutrinos do not change going from the tree level to the 1-loop-corrected level in the analysis of ref. [2].

The numerical evaluation of the model parameters and the masses of the light neutrinos in the case of the reduced parameterization is done similarly to the general case, described above. At the one-loop accuracy, our model will predict a vanishing mass for the lightest neutrino. If we use this value directly, the measured neutrino mass differences would lead to highly restricted values of the neutrino masses. In order to reduce the impact of a starting point to our analysis we allow the lightest neutrino to have a small non-vanishing mass, $m_{l_1}^{\text{in}}$. Using the central values of the experimental neutrino mass differences, we determine the largest initial value of the light neutrino masses m_l^{in} . Selecting the value of m_R , the value of m_D is evaluated from the seesaw relation, eq. (3.4). Since the vector \vec{a}_1 is fixed in this case, the tree-level neutrino mass matrix $M_\nu^{(0)}$ is fully defined as an input quantity at this point.

Solving the eigenvalue equation, we get the tree-level masses of the heaviest light neutrino $m_l^{(0)}$ and of the heavy neutrino $m_h^{(0)}$. The diagonalization matrix U_{tree} is constructed from a rotation matrix and a diagonal matrix of phases: $U_{\text{tree}} = U_{34}(\beta) \hat{U}_i$. Then one-loop corrections to the neutrino mass matrix are evaluated, and the parameters defining \vec{a}_2 enter into the further evaluation. Diagonalization of the 1-loop neutrino mass matrix $M_\nu^{(1)}$ is performed numerically with a unitary matrix U_{loop} as in the general parameterization case.

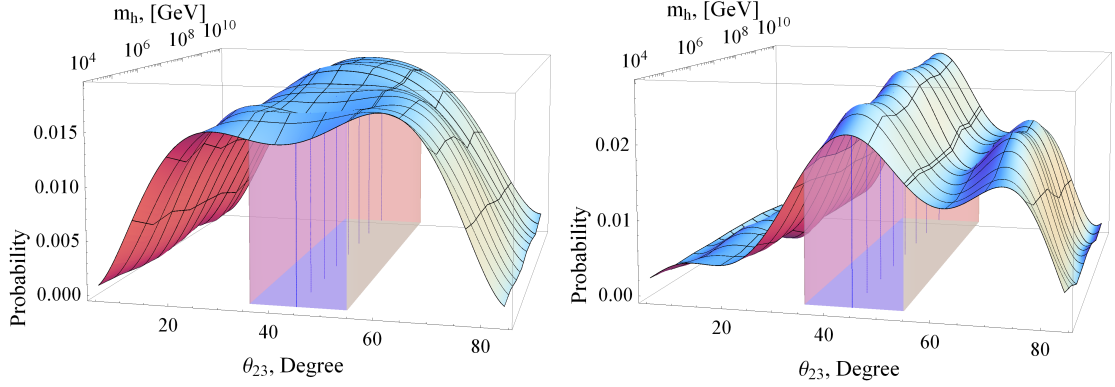


Figure 1. (Color online) The 3D histograms of $\theta_{23} = \theta_{\text{atm}}$ oscillation angles for $n_R = 1$. The left plot represents the case II of the Table 1. The right plot shows the case I. The filled boxes indicate the 3σ experimental boundaries [12], the blue vertical lines denote the experimental central value of θ_{23} .

After this procedure two light neutrinos have masses $m_{l_2}^{(1)}$ and $m_{l_3}^{(1)}$. For the calculation of Δm_{ij}^2 , we assume the mass of the lightest neutrino to be the input value $m_{l_1}^{\text{in}}$ with the justification, that it could be generated by a two-loop contribution. If the calculated mass differences

$$\Delta(m_{12}^{(1)})^2 = (m_{l_2}^{(1)})^2 - (m_{l_1}^{\text{in}})^2 \quad \text{and} \quad \Delta(m_{23}^{(1)})^2 = (m_{l_3}^{(1)})^2 - (m_{l_2}^{(1)})^2, \quad (3.10)$$

and the determined value of $\theta_{23} \approx \theta_{\text{atm}}$ (as described in the next paragraph) are compatible with the experimental data on oscillations, the model parameter set is kept. Otherwise, another set of input parameters $\{m_{l_1}^{\text{in}}, m_R, n, \phi, \alpha_{ij}, m_{H_{2,3}^0}\}$ is generated.

The results of the 1-loop corrected calculations are subject to the constraints from the experimental data on solar and atmospheric neutrino oscillations [12]. The neutrino mixing angles are determined from a factorization of the diagonalization matrix U_{loop} into terms where the PMNS matrix is included explicitly, following the ideas of ref. [13]. (The method is described in detail in appendix B.) For the case $n_R = 1$ it is possible to find exact analytical expressions for the mixing angles, Dirac and Majorana as well as non-physical phases. The oscillation angles are constrained by the experimental data in the sense that having a randomly generated set of the input parameters we derive the 1-loop corrected results and require that the estimated mixing angles are consistent with the experimental values in the 3σ range. It should be noted that this reduced parameterisation has only one non-vanishing neutrino mixing angle, namely, the atmospheric neutrino mixing angle θ_{atm} , because the vectors \vec{a}_1 and \vec{a}_2 have the same vanishing component ($a_{11} = a_{21} = 0$). In a more general case, all three oscillation angles are non-zero.

Distributions of the atmospheric oscillation angles for the cases I and II are shown in figure 1. The plot on the right shows the case I that is similar to the cases IIIa and IIIb. This distribution is narrower than the one obtained in the case II. The distribution for the case I has a well pronounced peak at around 45 degrees, while the probability to have an oscillation angle θ_{23} in the range from 25 to 65 degrees is nearly flat in the case II.

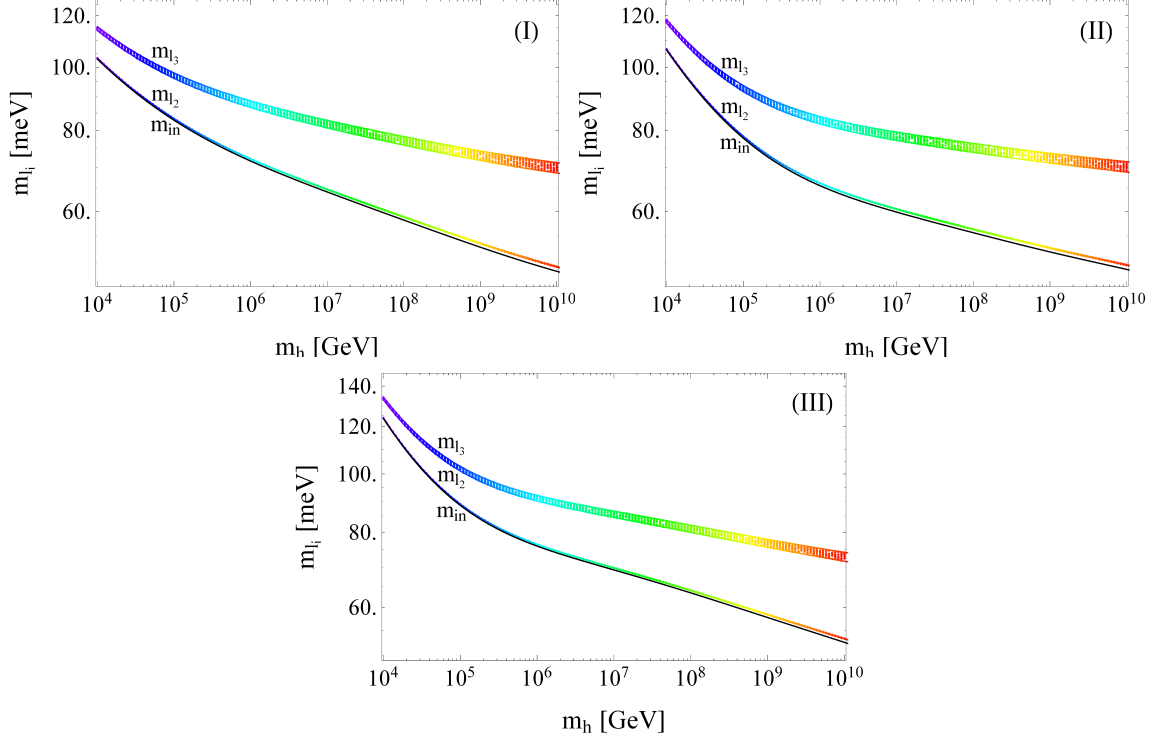


Figure 2. (Color online) Calculated masses of two light neutrinos as a function of the heaviest neutrino mass m_h for 3 different scenarios of Table 1 and $n_R = 1$. The black solid line represents the initial parameter m_{in} . Values of m_{l_2} and m_{l_3} are shown as a band where the color corresponds to the mass of the heavy neutrino m_h , given on the horizontal axis. This color code is also used in figures 3 and 4.

Since only two light neutrinos acquire mass in the case of $n_R = 1$ with 1-loop correction included, only normal ordering of neutrino masses is possible. Assuming that the lightest neutrino mass $m_{l_1} = 0$, we obtain a fixed spectrum of light neutrinos with $m_{l_2} = 8.7 \pm 0.3$ meV and $m_{l_3} = 50.6 \pm 2$ meV. However, the peak of the derived distribution of the oscillation angle θ_{atm} is shifted from the experimental value.

Figure 2 illustrates the neutrino mass spectrum for three different scenarios given in Table 1, assuming $m_{l_1} = m_{in} \neq 0$. All scenarios with $n_R = 1$ produce distributions of neutrino masses which differ from each other by value. In the $n_R = 2$ case (which is discussed below), all distributions look very similar, regardless of the scenario. The cases IIIa and IIIb have similar distributions, therefore no reference to the case 'a' or 'b' is given in the plot. The mass of the heaviest neutrino m_{l_3} reaches the highest value of 140 meV, when $m_h = 10^4$ GeV in the case III. The lowest value of $m_{l_3} = 70$ meV is obtained when $m_h = 10^{10}$ GeV in the cases I and II. The mass of the intermediate light neutrino m_{l_2} reaches the highest value of 121 meV, when $m_h = 10^4$ GeV in the case III, and the lowest value of $m_{l_2} = 48$ meV is reached when $m_h = 10^{10}$ GeV in the case I.

We observe a particular relationship between the values of the masses of light and heavy neutrinos. The calculated masses of the light neutrinos decrease, if the heavy neutrino mass

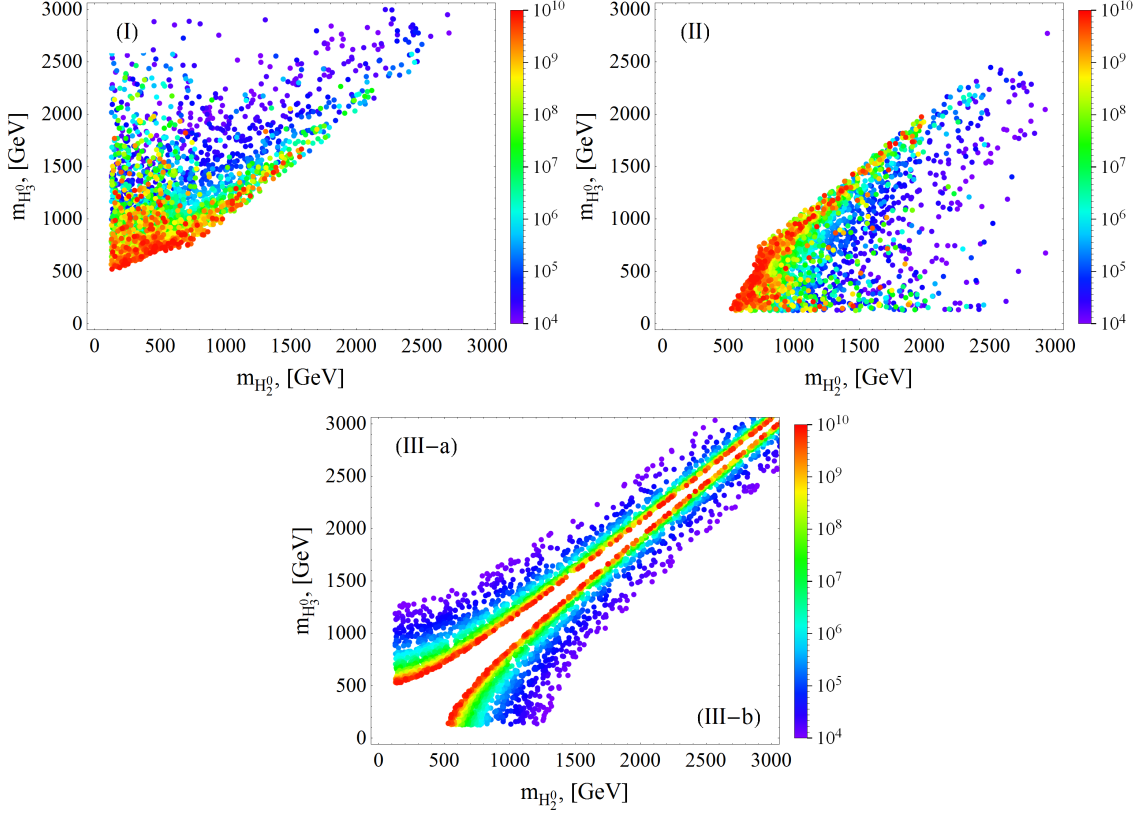


Figure 3. (Color online) Values of the neutral Higgs masses $m_{H_2^0}$ and $m_{H_3^0}$ as a function of the heaviest neutrino mass m_h for 4 different cases of Table 1 in the $n_R = 1$ model. The scale of the m_h values is shown on the right. The mass of the SM Higgs boson is fixed to $m_{H_1^0} = 125$ GeV.

increases. This dependence emerges from the relation of m_h to θ_{atm} . The value of m_{in} has to get lower as ν_R gets heavier in order to keep the most probable value of the oscillation angle within the experimental range of 3σ (this range is marked by boxes in figure 1). The narrow bands of the values of m_{l_2} and m_{l_3} are formed by a relatively high value of m_{in} and the restrictions on the mass differences. The results suggest that the lowest limit of the heavy neutrino mass is 10^4 GeV.

The allowed values of the Higgs masses (other than the SM Higgs) are illustrated in figure 3 as a function of the heavy singlet mass m_h . A band structure is formed according to the choice of vectors b (see Table 1) and the values of the free parameters n and ϕ , which are displayed in figure 4. When the mass of the heavy singlet m_h is increasing, the values of the Higgs masses tend to decrease in the 2HDM model cases I and II. This tendency is absent in the case III, where the allowed values of $m_{H_{2,3}^0}$ form a narrow band. Larger values of m_h make the masses of $m_{H_{2,3}^0}$ more similar, although their difference does not disappear. The range of the allowed masses of the heavier Higgs boson starts at 500 GeV.

The values of the Higgs masses displayed in figure 3 satisfy the experimental restrictions [1] of the oblique parameters S, T, and U, introduced by Peskin and Takeuchi [20].

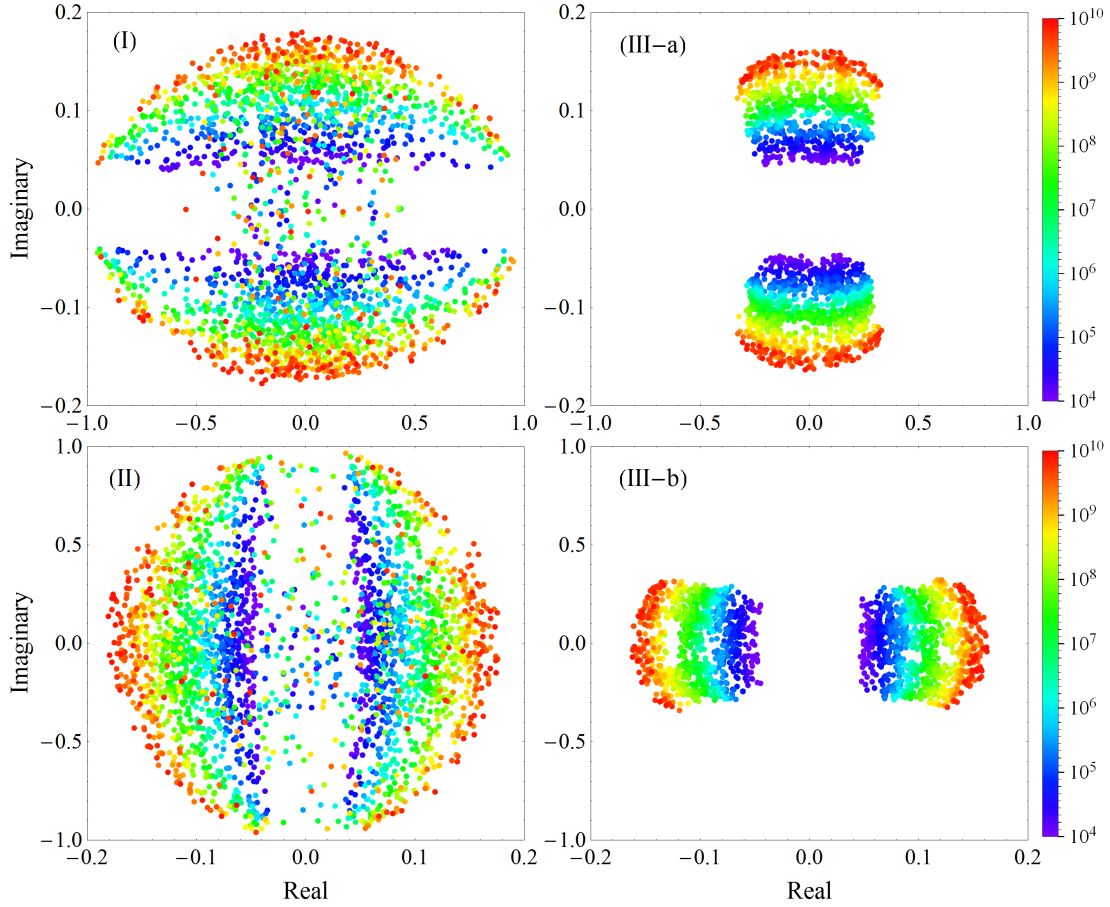


Figure 4. (Color online) Values of the free parameter a_{23} as a function of the heaviest neutrino mass m_h for the cases of Table 1 and $n_R = 1$. The scale of the m_h values is shown on the right.

These oblique parameters define combinations of observables that quantify deviations from the SM predictions. The experimental electroweak precision data yields values compatible with SM. We estimated these parameters using the algorithm implemented in the two Higgs doublet model calculator (2HDMC) [21] taking recent values of SM parameters [1]. The values of the oblique parameters are functions of masses of the three neutral ($m_{H_1^0}$, $m_{H_2^0}$, and $m_{H_3^0}$) and one charged (m_{H^\pm}) Higgs boson, and an additional factor that is related to the mixing angles of the neutral Higgses. This factor, defined as $\sin(\beta - \alpha)$ in ref. [21], equals to $\cos(\alpha_{13})$, $\cos(\alpha_{12})$, and 1 in the scenarios I, II, and III of Table 1, respectively. The estimated masses of the neutral Higgses and their mixing angles in our model are discussed above. The mass of the charged Higgs m_{H^\pm} is allowed to take values in the range of 126–3000 GeV. This defines the parameter space to estimate the values of S, T, and U. The calculated curves of S and U are mostly contained within the experimental bounds. The experimental limits on T are narrower as compared to the calculated distribution of T. A similar observation on the restrictive power of the oblique parameters is made in ref. [22]. These limits can be used to restrict the mass of the charged Higgs boson. If the

values of all three oblique parameters are compatible with the experimental data and the allowed value of m_{H^\pm} is within the studied range, the tested set of model parameters is kept. Otherwise, another set of parameters is generated. Using a stronger constraint of $U = 0$, which is also compatible with SM, about 35%–45% of the values in figure 3 satisfy the experimental restrictions [1]. In this case the majority of $m_{H_{2,3}^0}$ values fall in the range of 126–1500 GeV.

Distributions of the free parameter a_{23} , part of the neutrino couplings to the second Higgs doublet, are shown in Fig. 4. When the mass of the heavy singlet m_h increases, the absolute value of the parameter $|a_{23}|$ also increases. In case I the real part of a_{23} varies between -1 and 1 and the imaginary part varies from -0.18 to 0.18 . In case II real and imaginary parts behave like interchanged: the real part of a_{23} varies between -0.18 and 0.18 while imaginary part varies from -1 to 1 . In case III-a the real part of a_{23} varies between -0.34 and 0.34 while the imaginary part is restricted to the intervals $(0.04, 0.17)$ and $(-0.17, -0.04)$. In case III-b real and imaginary parts are exchanged like between cases I and II: the real part of a_{23} is restricted to the intervals $(0.04, 0.17)$ and $(-0.17, -0.04)$ while the imaginary part varies between -0.34 and 0.34 .

In summary, tuning the value of m_{in} , and restricting the light neutrino mass differences to the experimental central values within 1σ and the angle of oscillations $\theta_{23} = \theta_{\text{atm}}$ within 3σ , we determined the lowest limit of 10^4 GeV for the mass of the heavy neutrino singlet.

4 Results for the case $n_R = 2$

4.1 General parameterization

If we add two singlet fields ν_R to three left-handed neutrino fields ν_L , the radiative corrections give masses to all three light neutrinos. In the general case we parameterize

$$\Delta_i = \frac{\sqrt{2}}{v} \begin{pmatrix} m_{D_a} \vec{a}_i^\top \\ m_{D_b} \vec{b}_i^\top \end{pmatrix} \quad (4.1)$$

with 12 complex parameters of \vec{a}_i and \vec{b}_i , where $i = 1, 2$. We further assume that the vectors are normalized, $|\vec{a}_i| = |\vec{b}_i| = 1$. A specific set of the vector b values (3.1) was used for this study.

Numerical evaluation of the model parameters and the masses of the light neutrinos is performed in several steps. First, the neutrino mass matrix for tree level is constructed. The lightest neutrino is massless at tree level, $m_{l_1}^{(0)} = 0$. Taking $m_{l_1}^{\text{in}} = 0$, the masses of the other two light neutrinos, $m_{l_2}^{\text{in}}$ and $m_{l_3}^{\text{in}}$, are estimated from the experimental neutrino mass differences like for the case $n_R = 1$. Entries of the heavy neutrino mass matrix, $\hat{M}_R = \text{diag}(m_{R_1}, m_{R_2})$, are input parameters. The eigenvalue equation in the block form is:

$$U_{\text{tree}}^T M_\nu^{(0)} U_{\text{tree}} = U_{\text{tree}}^T \begin{pmatrix} 0_{3 \times 3} & m_{D_a} \vec{a}_1 & m_{D_b} \vec{b}_1 \\ m_{D_a} \vec{a}_1^\top & & \\ m_{D_b} \vec{b}_1^\top & & \hat{M}_R \end{pmatrix} U_{\text{tree}} = \begin{pmatrix} \hat{M}_l^{(0)} & 0 \\ 0 & \hat{M}_h^{(0)} \end{pmatrix}, \quad (4.2)$$

2HDM scenario	Normal hierarchy		Inverted hierarchy	
	Δ_1	Δ_2	Δ_1	Δ_2
I	$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b_{12} & b_{13} \end{pmatrix}$	$\begin{pmatrix} 0 & a_{22} & a_{23} \\ 0 & b_{22} & b_{23} \end{pmatrix}$	$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b_{12} & b_{13} \end{pmatrix}$	$\begin{pmatrix} a_{21} & a_{22} & 0 \\ b_{21} & 0 & b_{23} \end{pmatrix}$
II	$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b_{12} & b_{13} \end{pmatrix}$	$\begin{pmatrix} a_{21} & a_{22} & 0 \\ 0 & b_{22} & b_{23} \end{pmatrix}$	$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b_{12} & b_{13} \end{pmatrix}$	$\begin{pmatrix} 0 & a_{22} & a_{23} \\ b_{21} & 0 & b_{23} \end{pmatrix}$
III ^{a,b}	$\begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & b_{12} & b_{13} \end{pmatrix}$	$\begin{pmatrix} 0 & a_{22} & a_{23} \\ b_{21} & 0 & b_{23} \end{pmatrix}$	$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b_{12} & b_{13} \end{pmatrix}$	$\begin{pmatrix} a_{21} & a_{22} & 0 \\ b_{21} & 0 & b_{23} \end{pmatrix}$

Table 2. Textures of Dirac matrices used for calculations of the light neutrino mass spectra for normal and inverted hierarchies in four different cases of the 2HDM vectors b , listed in Table 1.

where $\hat{M}_h^{(0)} = \text{diag}(m_{h_1}^{(0)}, m_{h_2}^{(0)})$ with $m_{h_1}^{(0)} \leq m_{h_2}^{(0)}$. The values of m_{D_a} and m_{D_b} are evaluated through the seesaw equations, relating m_D , m_R on one side and m_l , m_h on the other side:

$$m_{D_a}^2 = m_{R_1} m_{l_2}^{\text{in}} \approx m_{h_1}^{(0)} m_{l_2}^{(0)}, \quad (4.3)$$

$$m_{D_b}^2 = m_{R_2} m_{l_3}^{\text{in}} \approx m_{h_2}^{(0)} m_{l_3}^{(0)}, \quad (4.4)$$

$$m_{R_i}^2 \approx (m_{h_i}^{(0)})^2, \quad i = 1, 2. \quad (4.5)$$

The diagonalization matrix for tree level $U_{\text{tree}} = U_{\text{egv}}^{\text{tree}} \hat{U}_\phi(\phi_i)$ is composed of the eigenmatrix of $M_\nu^{(0)\dagger} M_\nu^{(0)}$ (denoted by $U_{\text{egv}}^{\text{tree}}$), and a diagonal phase matrix \hat{U}_ϕ . At the second step we evaluate one-loop corrections to the neutrino mass matrix. Diagonalization of $M_\nu^{(1)}$ is performed with a unitary matrix $U_{\text{loop}} = U_{\text{egv}}^{\text{loop}} \hat{U}_\varphi(\varphi_i)$, where $U_{\text{egv}}^{\text{loop}}$ is the eigenmatrix of $M_\nu^{(1)\dagger} M_\nu^{(1)}$, and \hat{U}_φ is a phase matrix. This procedure yields masses for all three light neutrinos. If the calculated mass differences are compatible with the experimental data on neutrino oscillations, the model parameter set kept. Otherwise, another set of input parameters $\{m_{R_{1,2}}, \vec{a}_i, \vec{b}_i, m_{H_{2,3}^0}\}$ is generated.

The numerical analysis with the full set of parameters, constrained only to the experimental mass differences of the light neutrinos, suggests that heavy singlets should have mass greater than 100 GeV [7].

4.2 Reduced parameterization

Studying the influence of the randomly generated parameters we found that a reduced number of parameters is sufficient to fulfil the experimental criteria [12] of Δm_\odot^2 , Δm_{atm}^2 , θ_{12} , θ_{13} and θ_{23} . We selected several “textures” of Dirac matrices (i.e. the patterns of non-zero components), which allow the most accurate agreement to the experimental data. The texture of the matrix Δ_1 has the largest impact to the results of oscillation angles. There are 3 best versions of the Δ_1 textures:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b_{12} & b_{13} \end{pmatrix}, \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & b_{12} & b_{13} \end{pmatrix}, \text{ and } \begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & b_{12} & b_{13} \end{pmatrix}. \quad (4.6)$$

The textures of the matrix Δ_2 play a subdominant role for the results. In our calculations we tailored the textures to the 2HDM scenarios. As a result, the second texture of Δ_1 , having $a_{12} = b_{11} = 0$, was not used. The applied textures of Δ_1 and Δ_2 are listed in Table 2. All non-vanishing components can have complex values.

Studies of textures with one or two zero entries in the models with two heavy singlets have been reported in refs. [23–28]. Neither of those studies considered the SM extension by 2HDM. We obtain experimentally compatible neutrino oscillation values as in the mentioned references, but our textures are tuned to provide the model parameters in best agreement with the experimental data. It is worth mentioning that the textures listed in Table 2 may be a natural consequence of a certain flavor symmetry, as discussed in refs. [29–31].

The numerical evaluation of the model parameters and of the light neutrino masses, using the textures listed in Table 2, is done similarly to the general case. Having picked a mass for the lightest neutrino as an input value $m_{\text{in}} \equiv m_{l_1}^{\text{in}}$, the masses of the other two light neutrinos, $m_{l_2}^{\text{in}}$ and $m_{l_3}^{\text{in}}$, are estimated from the central values of experimental neutrino mass differences. Entries of the heavy neutrino mass matrix, $\hat{M}_R = \text{diag}(m_{R_1}, m_{R_2})$, are input parameters. The seesaw relations (4.3) give the values of m_{D_a} and m_{D_b} . The tree-level neutrino mass matrix $M_\nu^{(0)}$ is diagonalized, and U_{tree} is obtained. Then one-loop corrections to the neutrino mass matrix are calculated, and the parameters defining Δ_2 enter into the further evaluation. Diagonalization of the corrected neutrino mass matrix $M_\nu^{(1)}$ yields masses for all three light neutrinos. If the calculated mass differences are compatible with the experimental data on oscillations, including the three neutrino mixing angles, the model parameter set is kept. Otherwise, another set of input parameters $\{m_{\text{in}}, m_{R_{1,2}}, \vec{a}_i, \vec{b}_i, \alpha_{ij}, m_{H_{2,3}^0}\}$ is generated.

To increase the efficiency of random sampling, we chose the initial mass m_{in} in the range of 16–20 meV according to the agreement of the calculated results to the experimental values of the oscillation angles. Unlike the case $n_R = 1$, the oscillation angles are less affected by the value of m_{in} . Calculation of radiative corrections is done for the scenarios of the orthogonal complex vectors b listed in Table 1. The mass of the lightest neutral Higgs is fixed at $m_{H_1^0} = 125$ GeV but the masses of heavier neutral Higgses, $m_{H_2^0}$ and $m_{H_3^0}$, are generated randomly in the range from 126 to 3000 GeV.

Figure 5 shows the distributions of the oscillation angles for the scenario III using the best textures for Δ_1 and Δ_2 . Both normal and inverted neutrino mass hierarchies are shown. In the case of the inverted hierarchy the peaks of the distributions agree very well with the experimental bounds of all oscillation angles. In the case of the normal hierarchy, the most probable value of θ_{12} is slightly different from the experimental value. It is around 55 degrees, instead of the expected 33.4 degrees. However, if we extract θ_{12} from $\sin^2(2\theta_{12})$ we will find two solutions in the first quadrant. Following PDG [1], the experimental boundaries are in the regions 32.6–34.3 degrees and 55.7–57.4 degrees which agree with the peak of the atmospheric angle. In figure 5 the resulting values of θ_{13} are more localized as compared to the inverted hierarchy case. The distributions of θ_{23} agree with the experimental values in both hierarchies. The same tendency of the mixing angle distributions is seen in all three scenarios of the 2HDM basis. It should be noted that for

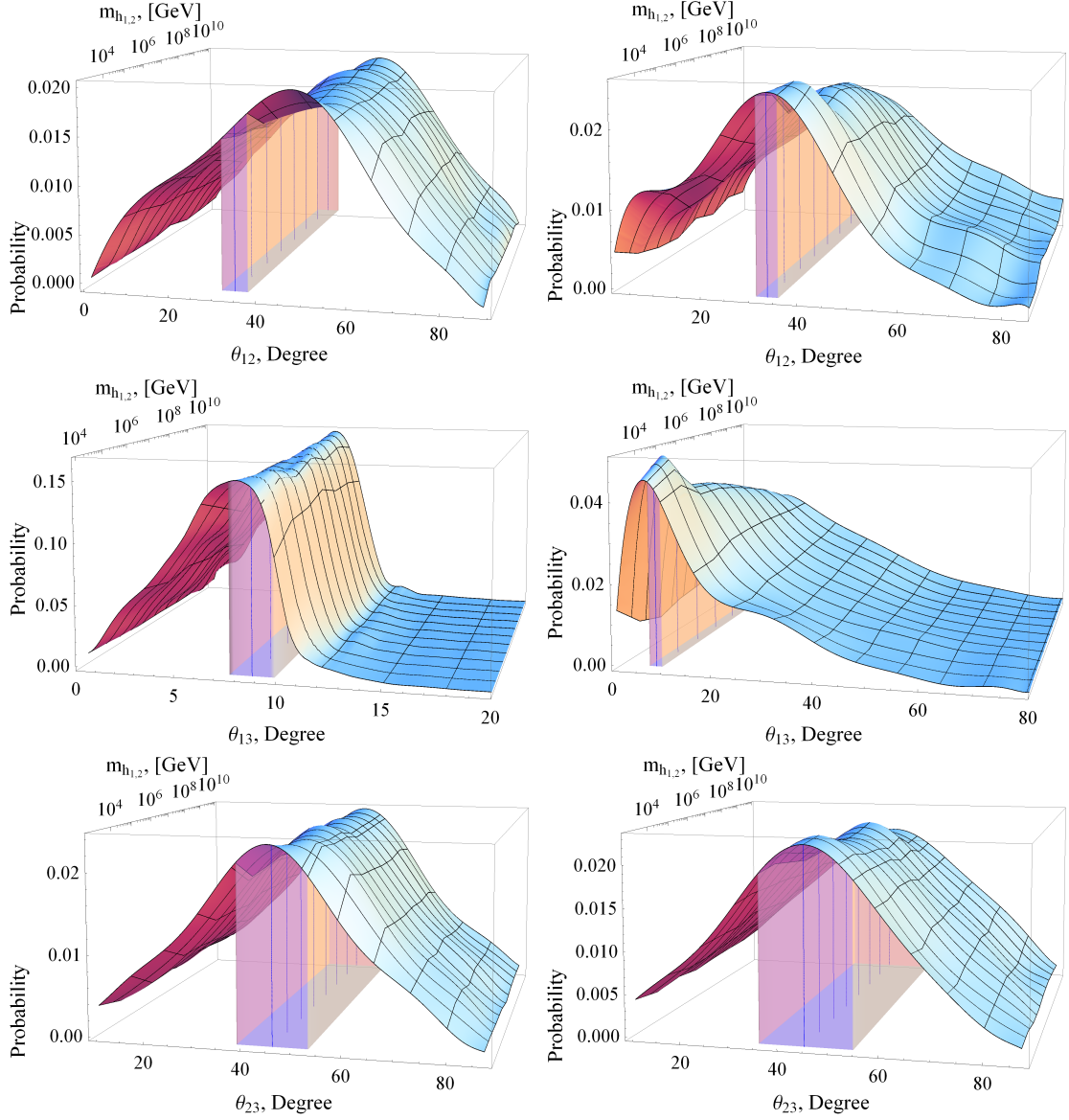


Figure 5. (Color online) The 3D histograms of the oscillation angles for the scenario III in the case $n_R = 2$. The plots on the left (right) correspond to the normal (inverted) hierarchy of the light neutrino masses. Filled boxes mark the applied experimental boundaries of 3σ , blue vertical lines mark the experimental central values.

the general parameterization case (where we have 12 complex parameters) all peaks of the oscillation angle distributions take values at approximately 45 degrees.

The neutrino mass spectrum is analyzed assuming that the masses of the heavy neutrinos are nearly equal, $m_{h_1} \approx m_{h_2}$ (the ratio m_{h_1}/m_{h_2} is 0.999). This simplifies the analysis and does not change the distributions of the light neutrino mass spectra in general terms (we can compare figure 6 to figure 2 in ref. [7]). The masses of the light neutrinos m_{l_i} are estimated when the masses $m_{h_{1,2}}$ vary in the range from 500 GeV to 10^{10} GeV, see

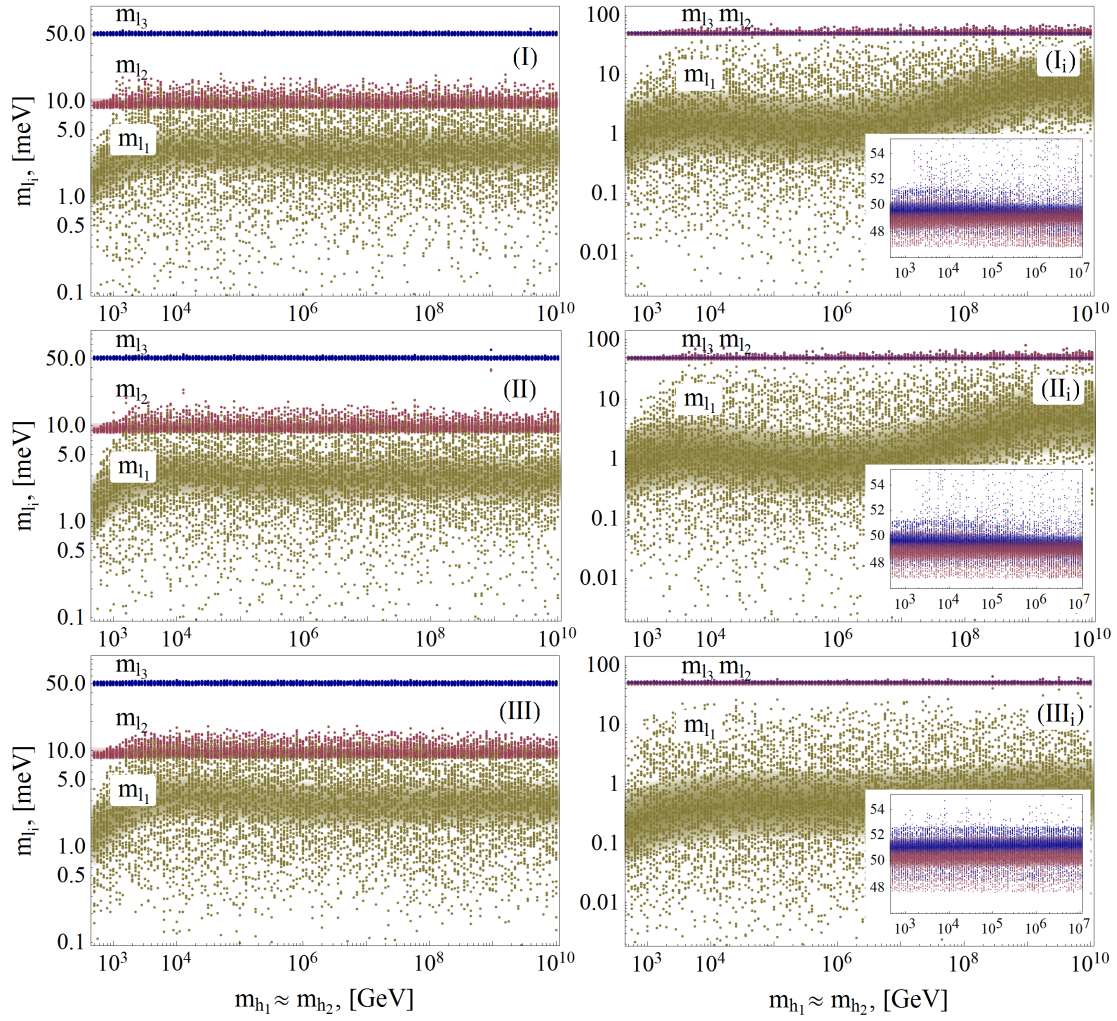


Figure 6. (Color online) Masses m_{l_i} of the light neutrinos as functions of the heaviest neutrino mass, $\max(m_{h_1}, m_{h_2})$, for the scenarios of 2HDM in the case of $n_R = 2$ (the scenarios IIIa and IIIb are very similar). The plots on the left represent the normal hierarchy, the plots on the right represent the inverted hierarchy of the light neutrinos. Wide bands indicate the area of the most frequent values of the scatter data. The nearly-degenerate masses m_{l_2} and m_{l_3} are shown separately in the lower right plots for the inverted hierarchy.

figure 6. Both normal and inverted hierarchies are shown. The dependency of the light neutrino masses on the values of the heavy neutrino masses is similar in all three scenarios. If normal hierarchy is assumed, the lightest neutrino mass m_{l_1} that is generated by the one-loop corrections varies from 0.01 to 20 meV. The most frequent values lie around 3 meV, when the mass of the heavy neutrinos is $m_{h_{1,2}} > 3000$ GeV. The mass m_{l_2} varies from 8 to 25 meV with the most frequent values at 10 meV. The mass of the heaviest light neutrino m_{l_3} is around 50 meV. If the inverted hierarchy is assumed, the range of m_{l_1} values is wider and varies from 0.001 to 40 meV. The most frequent values increase and

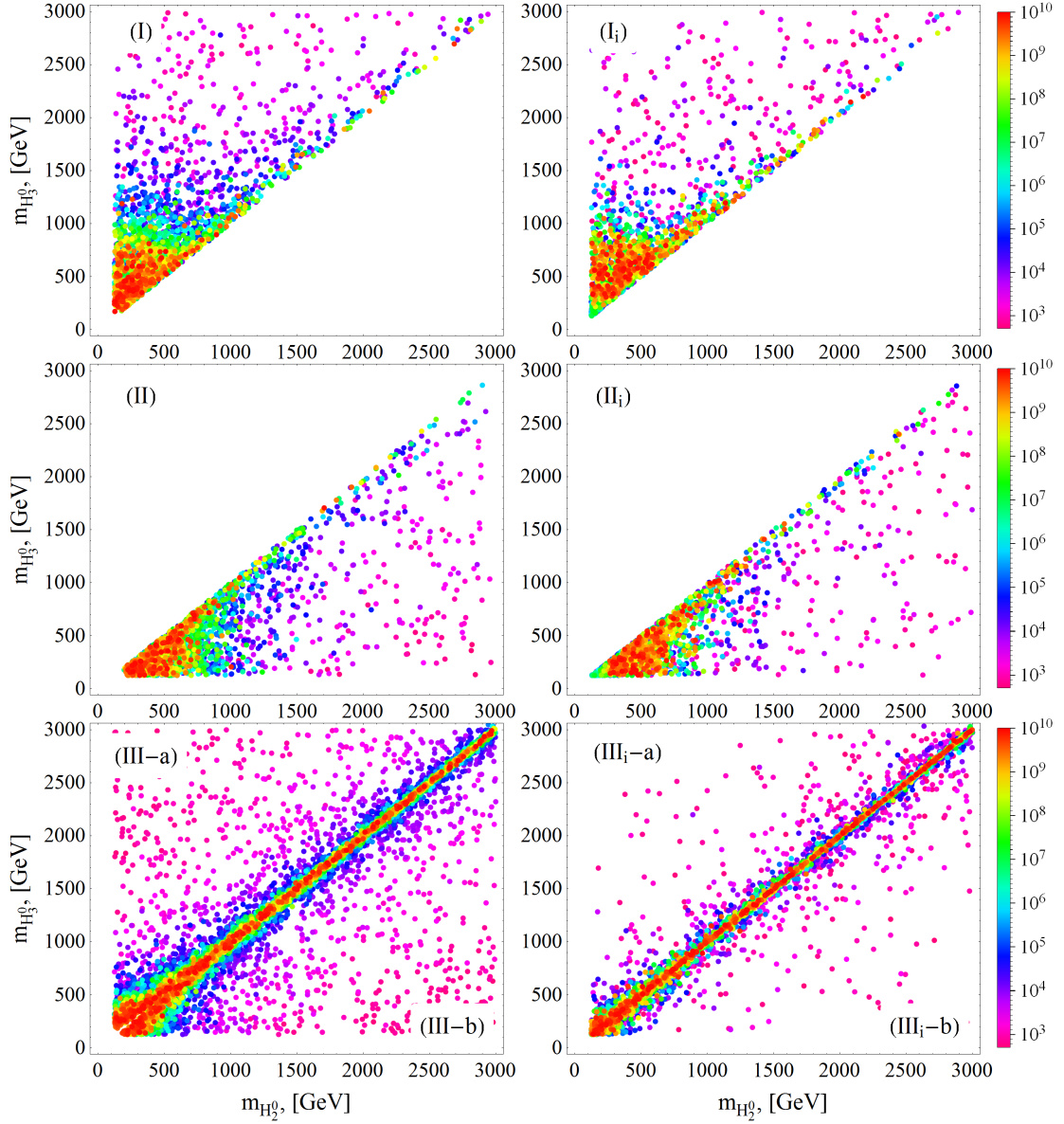


Figure 7. (Color online) Values of the free parameters $m_{H_2^0}$ and $m_{H_3^0}$ as functions of the heaviest neutrino mass $\max(m_{h_1}, m_{h_2})$ for the scenarios of 2HDM in the case $n_R = 2$. The plots on the left represent the normal hierarchy, the plots on the right represent the inverted hierarchy of the light neutrinos. The scale of the $\max(m_{h_1}, m_{h_2})$ values is shown on the right.

take values of 0.5–4 meV, depending on the masses of the heavy neutrinos. The values of m_{l_2} and m_{l_3} are nearly degenerate and vary from 47 to 55 meV. The most frequent values are in the range of 48–51 meV.

Figure 7 illustrates the allowed values of the Higgs masses depending on the masses of the heavy singlets $m_{h_{1,2}}$. Scenarios I and II are rather similar in dependencies, namely, an increase of the heavy singlet mass leads to the decrease of the Higgs masses. The mass of

the second Higgs boson tends to be different from the mass of the third Higgs boson. The scenario III has different dependencies. The heavy Higgs masses tend to be equal for large values of the heavy singlet masses, and tend to be independent of it.

The values of the Higgs masses displayed in figure 7 also satisfy the experimental restrictions [1] of the oblique parameters S, T, and U, as discussed in previous chapter. Using the stronger constraint of $U = 0$, about 40%–65% of the values shown in figure 7 satisfy the experimental restrictions. In this case the majority of $m_{H_{2,3}^0}$ values fall in the range of 126–1000 GeV.

The presented results are obtained using the tuned textures of Table 2. An alternative method to reduce the parameter space of vectors \vec{a} and \vec{b} could be used. For example, instead of setting an entire component (a_{ij} or b_{ij}) to zero, the parameter space could be limited restricting the values of these vectors to real numbers. A study of these textures will be reported in the future.

The most reduced parameterization of Dirac matrices, that still allows experimentally-compatible results of the light neutrino mass differences, has only 4 independent real parameters:

$$\Delta_1 = \frac{\sqrt{2}}{v} \begin{pmatrix} m_{D_a} a_1 & m_{D_a} \sqrt{1 - a_1^2} & 0 \\ 0 & m_{D_b} b_1 & m_{D_b} \sqrt{1 - b_1^2} \end{pmatrix}, \quad (4.7)$$

$$\Delta_2 = \frac{\sqrt{2}}{v} \begin{pmatrix} 0 & m_{D_a} a_2 & m_{D_a} \sqrt{1 - a_2^2} \\ m_{D_b} b_2 & m_{D_b} \sqrt{1 - b_2^2} & 0 \end{pmatrix}, \quad (4.8)$$

with $|a_i| \leq 1$ and $|b_i| \leq 1$, $i = 1, 2$. The neutrino oscillation angles evaluated in this strongly-reduced parameterization do not have the most-probable values in the experimentally determined range.

5 Summary

The seesaw mechanism is one of the most successful extensions of the SM which explains neutrino masses. Finite corrections to the neutrino mass matrix arise from one-loop diagrams mediated by a heavy neutrino. In our model the Higgs sector is constructed with two Higgs doublets and a CP-invariant Higgs potential which allows to distinguish four conditions for vectors b and thereby determine the scenarios (see Table 1) for numerical calculations. The SM Higgs mass is fixed to 125 GeV. By diagonalizing the neutrino mass matrix we obtain light neutrino masses and derive their oscillation angles. Sets of free parameters have been selected according to the distributions of oscillation angles within the experimental boundaries, when also the masses of the light neutrinos give the measured neutrino mass differences. In this paper we have studied two cases when one or two heavy neutrinos are added to the three light neutrinos. We refer to those cases as $n_R = 1$ and $n_R = 2$.

In the $n_R = 1$ case with a general parametrization of the Dirac matrices there are six free complex parameters. The numerical analysis suggests a lower limit of 830 GeV for the heavy singlet mass. However, the large number of free parameters makes it difficult to find

the correlations among them. By reducing the number of parameters we can study relations between them and the dependency on the heavy neutrino mass. The minimal reduction of free parameters, namely, $\vec{a}_1^\top = (0, a_{12}, a_{13})$ and $\vec{a}_2^\top = (a_{21}, a_{22}, a_{23})$, allows to estimate all three oscillation angles θ_{12} , θ_{13} and θ_{23} . The peaks of their distributions agree well with the experimental bounds. We presented results of a strongly reduced parameterization, $\vec{a}_1^\top = (0, 0, 1)$ and $\vec{a}_2^\top = (0, n, e^{i\phi}\sqrt{1-n^2})$. According to the chosen minimal set of the free parameters only the angle θ_{23} can be estimated. The calculated masses of the light neutrinos decrease, when the heavy neutrino mass increases. This dependency emerges from the relation of m_h and θ_{23} . Tuning the initial value m_{in} , and restricting the light neutrino mass differences to the experimental central values and the oscillation angle θ_{23} within 3σ , we determined for the mass of the heavy neutrino singlet a lowest limit of 10^4 GeV. When the mass of the heavy singlet m_h is increasing in the scenarios I and II of the 2HDM model, the allowed values of the non SM Higgs masses tend to decrease. This tendency is absent in the scenario III, where the Higgs boson masses $m_{H_{2,3}^0}$ get closer to each other, but stay different. We find a lower limit for the allowed values of the heavier Higgs boson mass of about 500 GeV. The values of the free parameters depend weakly on the mass of the heavy singlet m_h .

The general parametrization of the Dirac matrices in the $n_R = 2$ case has twelve complex parameters. The numerical analysis shows that the heavy singlets should have masses greater than 100 GeV. However, the most probable values of the neutrino oscillation angles are not in the experimentally determined range. We selected several textures of Dirac matrices, which allow the most accurate agreement to the experimental data. The texture of the matrix Δ_1 has the largest impact on the values of the oscillation angles while the textures of the matrix Δ_2 play a sub-dominant role. The used textures for normal and inverted neutrino mass hierarchies are listed in Table 2. The neutrino mass spectrum is analyzed assuming that the masses of the heavy neutrinos are nearly equal, $m_{h_1} \approx m_{h_2}$ (the ratio m_{h_1}/m_{h_2} is 0.999). The masses of the light neutrinos are estimated when the masses $m_{h_{1,2}}$ are greater than 500 GeV. The dependency of the light neutrino masses on the values of the heavy neutrino masses is similar in all three scenarios. An increase of the heavy singlet mass leads to the decrease of the non SM Higgs masses in the scenarios I and II. This tendency is absent in the scenario III, where the Higgs boson masses $m_{H_{2,3}^0}$ tend to be equal as the masses of the heavy singlets increase. The allowed values of $m_{H_{2,3}^0}$ can sample the entire range. Due to the large number of free parameters it is difficult to find correlations among them.

Our analysis has shown that the radiative corrections are quite sizeable and play an important role. They should be taken into account in the studies of the see-saw models. The studied case with one heavy singlet is a "toy" model because it is strongly restricted and does not provide all physical quantities. For example, the mass of the lightest neutrino is equal to zero, and we can evaluate only one oscillation angle. However, using this model it is possible to make some generalisations about the distributions of the Higgs masses and the correlations between the free parameters for models with a larger number of heavy singlets. The $n_R = 2$ case allows the calculation of all three masses of the light neutrinos with a reduced number of free parameters. The finding of textures which allow the most

accurate agreement of the oscillation angles to the experimental data could motivate some future models, for example, those based on the Abelian family symmetry or another discrete symmetry.

A Neutral Higgs mass eigenfields

Some features of formalism for the scalar sector of the multi-Higgs-doublet SM is given in ref. [2, 4]. Here we discuss the properties of the vectors b and give expressions for their calculation in the case of two Higgs doublets.

The physical neutral scalar mass eigenfields are expressed as

$$\phi_{b_k}^0 = \sqrt{2} \sum_{j=1}^{n_H} \text{Re}(b_{kj}^* \phi_j^0) = \frac{1}{\sqrt{2}} \sum_{j=1}^{n_H} (b_{kj}^* \phi_j^0 + b_{kj} \phi_j^{0*}), \quad (\text{A.1})$$

which are characterized by $2n_H$ unit vectors $b_k \in \mathbb{C}^{n_H}$ of dimensions $n_H \times 1$. In the matrix-vector notation, these eigenfields can be written as $\phi_{b_k}^0 = \sqrt{2} \text{Re}(b_k^\dagger \phi^0)$.

The orthonormality equations for the vectors are

$$\sum_{j=1}^{n_H} (\text{Re}(b_{kj}) \text{Re}(b_{k'j}) + \text{Im}(b_{kj}) \text{Im}(b_{k'j})) = \sum_{j=1}^{n_H} \text{Re}(b_{kj}^* b_{k'j}) = \delta_{b_k b_{k'}}; \quad (\text{A.2})$$

$$\sum_{k=1}^{2n_H} \text{Re}(b_{kj}) \text{Re}(b_{kj'}) = \sum_{k=1}^{2n_H} \text{Im}(b_{kj}) \text{Im}(b_{kj'}) = \delta_{jj'}; \quad (\text{A.3})$$

$$\sum_{k=1}^{2n_H} \text{Re}(b_{kj}) \text{Im}(b_{kj'}) = \sum_{k=1}^{2n_H} b_{kj} b_{kj'} = 0. \quad (\text{A.4})$$

The vectors b_k and $b_{k'}$ indicate two different states $\phi_{b_k}^0$ and $\phi_{b_{k'}}^0$, and indices j and j' indicate two different components of the vectors b .

The neutral Goldstone boson $G^0 = \phi_{G^0}^0$ corresponds to the vector b_{G^0} with the components $(b_{G^0})_j = i v_j / v$ [2, 4], where $v = \left(|v_1|^2 + |v_2|^2 + \dots + |v_{n_H}|^2 \right)^{1/2} = 2m_W / g$. In the case of only two Higgs doublets, and due to the rotation of the Higgs fields to make the vacuum expectation value a feature of the SM Higgs field, the vector b_{G^0} equals

$$b_{G^0} = \begin{pmatrix} i \\ 0 \end{pmatrix}. \quad (\text{A.5})$$

Physical Higgs fields $\phi_{b_k \neq G^0}^0$ must be orthogonal to the Goldstone field G^0 which follows from (A.2). This leads to the condition

$$\sum_{j=1}^{n_H} \text{Re} \left(-\frac{i v_j}{v} b_{kj}^* \right) = \frac{1}{v} \sum_{j=1}^{n_H} \text{Im} (v_j b_{kj}^*) = \sum_{j=1}^{n_H} \text{Re} (b_{G^0 j} b_{kj}^*) = 0. \quad (\text{A.6})$$

To study the unit vectors b , introduced in eq. (A.1) (which is the same as eq. (2.2) in the text) and corresponding to the Higgs fields other than the Goldstone boson G^0 , lets

define them in the following form:

$$b_1 = \begin{pmatrix} b_{11} \\ b_{12} \end{pmatrix}, \quad b_2 = \begin{pmatrix} b_{21} \\ b_{22} \end{pmatrix}, \quad b_3 = \begin{pmatrix} b_{31} \\ b_{32} \end{pmatrix}. \quad (\text{A.7})$$

From the orthogonality relations (A.2 - A.4) and due to the fixed value of b_{G^0} (A.5) it is possible to write the orthogonality equations for vector components in the following manner:

$$b_{11}, b_{21}, b_{31} \in \mathbb{R}; \quad b_{12}, b_{22}, b_{32} \in \mathbb{C}; \quad (\text{A.8})$$

$$b_{k1}^2 + |b_{k2}|^2 = 1; \quad (\text{A.9})$$

$$b_{k1}b_{k'1} + \text{Re}(b_{k2}^*b_{k'2}) = 0; \quad (\text{A.10})$$

$$\sum_{k=1}^3 b_{k2}^2 = \sum_{k=1}^3 b_{k1}b_{k2} = 0; \quad (\text{A.11})$$

$$\sum_{k=1}^3 b_{k1}^2 = \sum_{k=1}^3 [\text{Re}(b_{k2})]^2 = \sum_{k=1}^3 [\text{Im}(b_{k2})]^2 = 1. \quad (\text{A.12})$$

By choosing b_{31} , b_{21} and $\text{Re}(b_{32})$ as input variables, it is possible to express the other components of the vectors b by those variables by solving the equations (A.8 - A.12). Introducing three sign-parameters $s_{32\text{im}}$, s_{11} , and s_{22} (they can take values ± 1), we can write

$$\text{Im}(b_{32}) = s_{32\text{im}} \sqrt{1 - b_{31}^2 - [\text{Re}(b_{32})]^2}; \quad (\text{A.13})$$

$$b_{11} = s_{11} \sqrt{1 - b_{31}^2 - b_{21}^2}; \quad (\text{A.14})$$

$$b_{\text{comb}} \equiv \frac{b_{31}b_{21}\text{Re}(b_{32}) + s_{22}|b_{11}||\text{Im}(b_{32})|}{b_{31}^2 - 1}; \quad (\text{A.15})$$

$$p_{22} \equiv \begin{cases} -\text{Sg}(b_{31})\text{Sg}(b_{21})\text{Sg}(\text{Im}(b_{32})), & \text{if } |\text{Re}(b_{32})| \leq \sqrt{\frac{b_{31}^2 b_{21}^2}{1 - b_{21}^2}}, \\ s_{22}\text{Sg}(\text{Re}(b_{32}))\text{Sg}(\text{Im}(b_{32})), & \text{otherwise}; \end{cases} \quad (\text{A.16})$$

$$b_{22} = b_{\text{comb}} + ip_{22} \sqrt{1 - b_{21}^2 - b_{\text{comb}}^2}, \quad (\text{A.17})$$

$$b_{12} = -\frac{1}{b_{11}}(b_{31}b_{32} + b_{21}b_{22}). \quad (\text{A.18})$$

We introduced two intermediate parameters b_{comb} and p_{22} , and $\text{Sg}(x)$ is the sign function

$$\text{Sg}(x) = \begin{cases} -1, & x < 0, \\ 1, & x \geq 0. \end{cases} \quad (\text{A.19})$$

It is worth mentioning that the solutions for the parameter values, given by the equations (A.13 - A.17), were obtained assuming $b_{21}, b_{31} \neq \pm 1$. According to the orthogonality relations (A.8 - A.12) the free scale parameters vary in the following ranges: $|b_{31}| < 1$, $|b_{21}| < \sqrt{1 - b_{31}^2}$, and $|\text{Re}(b_{32})| \leq \sqrt{1 - b_{31}^2}$. The extreme values of ± 1 for the parameters

b_{21} and b_{31} could be obtained by the index permutation of the vectors b_k (for example, $b_{21} = 1$ can be obtained by swapping the values of $b_{11} = 1$ and b_{12} with those of b_{21} and b_{22}).

Equations (A.13 - A.17) give 8 different solutions for the vectors b , corresponding to two possible values of the sign-parameters s_x ($x = 32\text{im}, 11$, and 22). However, due to the structure of the one-loop corrections (2.22), only 4 different solutions of those equations are important, since the sign of b_{11} (i.e. the value of s_{11}) does not change the values of the light neutrino masses.

The expressions of eqs. (A.13 - A.17) are significantly simpler, if some input parameters are equal to zero. This can lead to further simplification after introducing trigonometric functions. Let us study the case, when $\text{Re}(b_{32}) = 0$. Defining $b_{31} = \sin(\alpha_{13})$, $b_{21} = \sin(\alpha_{12})\cos(\alpha_{13})$, and taking $s_{32\text{im}} = s_{11} = s_{22} = 1$, we obtain the following parametric values of the vectors b :

$$b_{G^0} = \begin{pmatrix} i \\ 0 \end{pmatrix}, \quad b_1 = \begin{pmatrix} c_{12}c_{13} \\ -s_{12} - ic_{12}s_{13} \end{pmatrix}, \quad b_2 = \begin{pmatrix} s_{12}c_{13} \\ c_{12} - is_{12}s_{13} \end{pmatrix}, \quad b_3 = \begin{pmatrix} s_{13} \\ ic_{13} \end{pmatrix}, \quad (\text{A.20})$$

where $c_{ij} \equiv \cos(\alpha_{ij})$ and $s_{ij} \equiv \sin(\alpha_{ij})$.

B Neutrino oscillation angles

Neutrino oscillation angles are introduced using the tree-level neutrino mass diagonalization matrix U_{loop} and factorizing it to contain the ordinary Pontecorvo-Maki-Nakagawa-Sakata (PMNS) neutrino mixing matrix. We introduce the formalism by discussing the 3×3 neutrino mixing case, where the relationships are simpler. Then we discuss the cases that are used in current paper, namely, $(3+1) \times (3+1)$ and $(3+2) \times (3+2)$ neutrino mixing.

The simplest case (3×3) considers only the SM neutrinos. It is discussed in ref. [32] in a slightly different notation of the matrix elements. Factorization of the rotation matrix with the PMNS matrix included explicitly in the case $3+3$ is discussed in ref. [13]. Here we give formulas for the intermediate cases.

The neutrino masses and the mixing angles are predicted from a given neutrino mass matrix (the “top-down” method). Exact analytical expressions for the mixing angles, Dirac and Majorana phases and formulas for the non-physical phases can be given for the 3- and 4-dimensional cases. Only numerical solutions are possible in the case of 2 additional neutrinos (the 5-dimensional case).

The 3-dimensional case

First we parameterize the neutrino diagonalisation matrix by including the PMNS mixing matrix for the 3×3 mixing. The neutrino mass matrix can be diagonalised by a unitary transformation U , obtained by the singular value decomposition method, see eq. (2.13). Lets denote the matrix elements in the following way:

$$U^{(3 \times 3)} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}. \quad (\text{B.1})$$

This matrix could be factorized into three terms

$$U^{(3 \times 3)} = \hat{U}_\phi^{(3)} \cdot U_{\text{PMNS}} \cdot \hat{U}_\kappa^{(3)}, \quad (\text{B.2})$$

where U_{PMNS} is the standard PMNS mixing matrix for Dirac neutrinos:

$$\begin{aligned} U_{\text{PMNS}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \cdot \begin{pmatrix} c_{13} & 0 & \hat{s}_{13}^* \\ 0 & 1 & 0 \\ -\hat{s}_{13} & 0 & c_{13} \end{pmatrix} \cdot \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_{12}c_{13} & c_{13}s_{12} & \hat{s}_{13}^* \\ -c_{23}s_{12} - c_{12}\hat{s}_{13}s_{23} & c_{12}c_{23} - s_{12}\hat{s}_{13}s_{23} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}c_{23}\hat{s}_{13} & -c_{23}s_{12}\hat{s}_{13} - c_{12}s_{23} & c_{13}c_{23} \end{pmatrix}. \end{aligned} \quad (\text{B.3})$$

We used abbreviations $c_{ij} \equiv \cos \theta_{ij}$ and $\hat{s}_{ij} \equiv e^{i\delta_{ij}} \sin \theta_{ij}$, where θ_{ij} and δ_{ij} are the rotation angle and the phase angle, respectively.

The two diagonal phase matrices are defined as

$$\hat{U}_\phi^{(3)} = \begin{pmatrix} e^{i\phi_1} & 0 & 0 \\ 0 & e^{i\phi_2} & 0 \\ 0 & 0 & e^{i\phi_3} \end{pmatrix}, \quad (\text{B.4})$$

$$\hat{U}_\kappa^{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\kappa_1/2} & 0 \\ 0 & 0 & e^{i\kappa_2/2} \end{pmatrix}. \quad (\text{B.5})$$

There are 9 parameters: 3 mixing angles (θ_{12} , θ_{13} , θ_{23}); 1 Dirac phase δ_{13} ; 2 Majorana phases κ_1 and κ_2 ; and the matrix $\hat{U}_\phi^{(3)}$ containing 3 non-physical and unmeasurable phases ϕ_i ($i = 1, 2, 3$).

Comparing eqs. (B.1) and (B.2) we can find the relations between the elements of the rotation matrix in a general form and in the factorized form:

$$\theta_{13} = \arcsin(|x_3|), \quad \theta_{23} = \arcsin\left(\frac{|y_3|}{\sqrt{1-|x_3|^2}}\right), \quad \theta_{12} = \arcsin\left(\frac{|x_2|}{\sqrt{1-|x_3|^2}}\right), \quad (\text{B.6})$$

$$\frac{\kappa_1}{2} = \arg(x_2) - \arg(x_1), \quad \frac{\kappa_2}{2} = \arg(x_3) - \arg(x_1) + \delta_{13}, \quad (\text{B.7})$$

$$\phi_1 = \arg(x_1), \quad \phi_2 = \arg(x_1) - \arg(x_3) + \arg(y_3) - \delta_{13}, \quad (\text{B.8})$$

$$\phi_3 = \arg(x_1) - \arg(x_3) + \arg(z_3) - \delta_{13}, \quad (\text{B.9})$$

$$\delta_{13} = \arg(x_2) - \arg(x_3) + \arg(y_3) + i \ln \left(\frac{y_2(1-|x_3|^2) + x_2 x_3^* y_3}{|x_1||z_3|} \right). \quad (\text{B.10})$$

These relations are obtained comparing eq. (B.2) with the following matrix elements from eq. (B.1), forming the upper-triangular matrix: x_1 , x_2 , x_3 , y_2 , y_3 , and z_3 . Other (identical) solutions are possible, using the matrix elements y_1 , z_1 , and z_2 .

4-dimensional case

If there is one additional neutrino, decomposition of the neutrino mass diagonalization matrix into factors including the PMNS neutrino mixing matrix is more complicated. Lets define 2-dimensional rotation matrices in the 4-dimensional complex space, similarly to ref. [13],

$$\begin{aligned}
R_{12}^{(4)} &= \begin{pmatrix} c_{12} & s_{12} & 0 & 0 \\ -s_{12} & c_{12} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & R_{13}^{(4)} &= \begin{pmatrix} c_{13} & 0 & \hat{s}_{13}^* & 0 \\ 0 & 1 & 0 & 0 \\ -\hat{s}_{13} & 0 & c_{13} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
R_{23}^{(4)} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{23} & s_{23} & 0 \\ 0 & -s_{23} & c_{23} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & R_{14}^{(4)} &= \begin{pmatrix} c_{14} & 0 & 0 & \hat{s}_{14}^* \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\hat{s}_{14} & 0 & 0 & c_{14} \end{pmatrix}, \\
R_{24}^{(4)} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{24} & 0 & \hat{s}_{24}^* \\ 0 & 0 & 1 & 0 \\ 0 & -\hat{s}_{24} & 0 & c_{24} \end{pmatrix}, & R_{34}^{(4)} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_{34} & \hat{s}_{34}^* \\ 0 & 0 & -\hat{s}_{34} & c_{34} \end{pmatrix}, \tag{B.11}
\end{aligned}$$

and phase matrices: $\hat{U}_\phi^{(4)} = \text{diag}(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}, e^{i\phi_4})$ and $\hat{U}_\kappa^{(4)} = \text{diag}(1, e^{i\kappa_1/2}, e^{i\kappa_2/2}, 1)$. Note that a shorter notation can be used to define the elements of the rotation matrices:

$$[R_{jk}^{(4)}]_a^b = \delta_a^b + (c_{jk} - 1)(\delta_a^j \delta_j^b + \delta_a^k \delta_k^b) + \hat{s}_{jk}^* \delta_a^j \delta_k^b - \hat{s}_{jk} \delta_a^k \delta_j^b, \tag{B.12}$$

where δ_a^b equals 1, when $a = b$, or 0, otherwise. This notation is not restricted to the 4-dimensional case.

The unitary matrix $U^{(4 \times 4)}$ is parameterized by

$$U^{(4 \times 4)} = \hat{U}_\phi^{(4)} \cdot (R_{34}^{(4)} R_{24}^{(4)} R_{14}^{(4)}) \cdot (R_{23}^{(4)} R_{13}^{(4)} R_{12}^{(4)}) \cdot \hat{U}_\kappa^{(4)}, \tag{B.13}$$

with the PMNS matrix defined by a product of three rotation matrices:

$$\begin{pmatrix} U_{\text{PMNS}} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} = (R_{23}^{(4)} R_{13}^{(4)} R_{12}^{(4)}). \tag{B.14}$$

There are 16 parameters in this case, namely: 6 mixing angles $(\theta_{12}, \theta_{13}, \theta_{23}, \theta_{14}, \theta_{24}, \theta_{34})$; 1 Dirac phase δ_{13} ; 2 Majorana phases κ_1 and κ_2 ; 3 phases $\delta_{14}, \delta_{24}, \delta_{34}$; and the matrix $\hat{U}_\phi^{(4)}$ containing 4 phases ϕ_i ($i = 1, 2, 3, 4$).

For our model with $n_R = 1$ we calculate the diagonalization matrix U_{loop} numerically. Defining its elements as

$$U^{(4 \times 4)} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ t_1 & t_2 & t_3 & t_4 \end{pmatrix} \tag{B.15}$$

and comparing to eq. (B.13) we find the relations:

$$\theta_{12} = \arcsin\left(\frac{|x_2|}{\sqrt{b}}\right), \quad \theta_{13} = \arcsin\left(\frac{|x_3|}{\sqrt{a}}\right), \quad \theta_{23} = \arcsin\left(\frac{|d|}{\sqrt{b}\sqrt{c}}\right), \quad (\text{B.16})$$

$$\theta_{14} = \arcsin(|x_4|), \quad \theta_{24} = \arcsin\left(\frac{|y_4|}{\sqrt{a}}\right), \quad \theta_{34} = \arcsin\left(\frac{|z_4|}{\sqrt{c}}\right), \quad (\text{B.17})$$

$$\phi_1 = \arg(x_1), \quad \phi_4 = \arg(t_4), \quad (\text{B.18})$$

$$\begin{aligned} \phi_2 &= \arg(x_1) - \arg(x_2) \\ &- i \ln \left(\frac{a b y_2 + x_2 x_4^* y_4 \sqrt{b} \sqrt{a - |x_3|^2} + d |a| x_2 x_3^* \sqrt{a - |y_4|^2} / (a \sqrt{c})}{\sqrt{a} \sqrt{b c} - |d|^2 \sqrt{b - |x_2|^2}} \right), \end{aligned} \quad (\text{B.19})$$

$$\phi_3 = \phi_2 + i \ln \left(\frac{d |a| \sqrt{a} \sqrt{b c} - |d|^2 \sqrt{a - |x_3|^2} \sqrt{c - |z_4|^2} / |d|}{a^2 \sqrt{b c} z_3 + a \sqrt{b} \sqrt{c} x_3 x_4^* z_4 \sqrt{a - |y_4|^2} + d |a| y_4^* z_4 \sqrt{a - |x_3|^2}} \right), \quad (\text{B.20})$$

$$\delta_{13} = \arg(x_1) - \arg(x_3) - \phi_2 - i \ln \left(\frac{d |a|}{a |d|} \right), \quad (\text{B.21})$$

$$\delta_{14} = \phi_1 - \arg(x_4), \quad \delta_{24} = \phi_2 - \arg(y_4), \quad \delta_{34} = \arg(z_4) - \phi_3, \quad (\text{B.22})$$

$$\frac{\kappa_1}{2} = \arg(x_2) - \phi_1, \quad \frac{\kappa_2}{2} = -\phi_2 - i \ln \left(\frac{d |a|}{a |d|} \right), \quad (\text{B.23})$$

where:

$$\begin{aligned} a &= 1 - |x_4|^2, & b &= 1 - |x_3|^2 - |x_4|^2, \\ c &= 1 - |x_4|^2 - |y_4|^2, & d &= a y_3 + x_3 x_4^* y_4. \end{aligned} \quad (\text{B.24})$$

Because of the discontinuous nature of the square root function in the complex plane, $\sqrt{xy} \neq \sqrt{x}\sqrt{y}$ in general. Therefore a simplification of the above expressions is limited.

Due to its origin and the relations between the elements, the expressions do not contain all entries of the rotation matrix $U^{(4 \times 4)}$, defined in eq. (B.15). These relations are obtained comparing eq. (B.13) with the following matrix elements from eq. (B.15), forming the upper-triangular matrix: $x_1, x_2, x_3, x_4, y_2, y_3, y_4, z_3, z_4$, and t_4 . Other (identical) solutions are possible using the matrix elements z_1, z_2, t_1, t_2 , and t_3 .

5-dimensional case

To introduce factorization containing the PMNS neutrino mixing matrix in the $3 + 2$ case, we first define the rotation matrices in the 5-dimensional complex space, similarly to ref. [13]

$$R_{12}^{(5)} = \begin{pmatrix} c_{12} & s_{12} & 0 & 0 & 0 \\ -s_{12} & c_{12} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad R_{13}^{(5)} = \begin{pmatrix} c_{13} & 0 & \hat{s}_{13}^* & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -\hat{s}_{13} & 0 & c_{13} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{aligned}
R_{23}^{(5)} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & c_{23} & s_{23} & 0 & 0 \\ 0 & -s_{23} & c_{23} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, & R_{14}^{(5)} &= \begin{pmatrix} c_{14} & 0 & 0 & \hat{s}_{14}^* & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -\hat{s}_{14} & 0 & 0 & c_{14} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\
R_{24}^{(5)} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & c_{24} & 0 & \hat{s}_{24}^* & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -\hat{s}_{24} & 0 & c_{24} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, & R_{34}^{(5)} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & c_{34} & \hat{s}_{34}^* & 0 \\ 0 & 0 & -\hat{s}_{34} & c_{34} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\
R_{15}^{(5)} &= \begin{pmatrix} c_{15} & 0 & 0 & 0 & \hat{s}_{15}^* \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\hat{s}_{15} & 0 & 0 & 0 & c_{15} \end{pmatrix}, & R_{25}^{(5)} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & c_{25} & 0 & 0 & \hat{s}_{25}^* \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -\hat{s}_{25} & 0 & 0 & c_{25} \end{pmatrix}, \\
R_{35}^{(5)} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & c_{35} & 0 & \hat{s}_{35}^* \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\hat{s}_{35} & 0 & c_{35} \end{pmatrix}, & R_{45}^{(5)} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & c_{45} & \hat{s}_{45}^* \\ 0 & 0 & 0 & -\hat{s}_{45} & c_{45} \end{pmatrix},
\end{aligned}$$

and the phase matrices:

$$\hat{U}_\phi^{(5)} = \text{diag} \left(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}, e^{i\phi_4}, e^{i\phi_5} \right), \quad (\text{B.25})$$

$$\hat{U}_\kappa^{(5)} = \text{diag} \left(1, e^{i\kappa_1/2}, e^{i\kappa_2/2}, 1, 1 \right). \quad (\text{B.26})$$

The unitary matrix $U^{(5 \times 5)}$ is parameterized by

$$U^{(5 \times 5)} = U_\phi^{(5)} \cdot \left(R_{45}^{(5)} R_{35}^{(5)} R_{25}^{(5)} R_{15}^{(5)} \right) \cdot \left(R_{34}^{(5)} R_{24}^{(5)} R_{14}^{(5)} \right) \cdot \left(R_{23}^{(5)} R_{13}^{(5)} R_{12}^{(5)} \right) \cdot U_\kappa^{(5)} \quad (\text{B.27})$$

with the inclusion of the PMNS matrix

$$\begin{pmatrix} U_{\text{PMNS}} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} = \left(R_{23}^{(5)} R_{13}^{(5)} R_{12}^{(5)} \right). \quad (\text{B.28})$$

There are 25 parameters in the 5-dimensional case: 10 mixing angles ($\theta_{12}, \theta_{13}, \theta_{23}, \theta_{14}, \theta_{24}, \theta_{34}, \theta_{15}, \theta_{25}, \theta_{35}, \theta_{45}$); 1 Dirac phase δ_{13} ; 2 Majorana phases κ_1 and κ_2 ; 7 phases $\delta_{14}, \delta_{24}, \delta_{34}, \delta_{15}, \delta_{25}, \delta_{35}, \delta_{45}$; and the matrix $\hat{U}_\phi^{(5)}$ containing 5 phases ϕ_i , $i = 1, \dots, 5$. Only numerical solutions for the parameters are possible.

A simplification is possible in our analysis. According to the structure of the diagonalisation matrix U_{loop} in the 4- or 5-dimensional cases, the 3×3 sub-matrix in the top-left corner is dominant. This sub-matrix corresponds to the matrix U_{PMNS} . In the numerical calculation, it suffices to use the expressions B.6 in order to estimate the oscillation angles

θ_{12}, θ_{13} and θ_{23} , in the cases of $n_R = 1$ and $n_R = 2$. There is a numerical precision difference between the approximated angle values and the values obtained using the expressions (B.16) in the 4-dimensional case or the numerical solutions in the 5-dimensional case. The approximation speeds up the calculations significantly.

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References

- [1] **Particle Data Group** Collaboration, K. Olive et al., *Review of Particle Physics*, *Chin.Phys.* **C38** (2014) 090001.
- [2] W. Grimus and H. Neufeld, *Radiative Neutrino Masses in an $SU(2) \times U(1)$ Model*, *Nucl.Phys.* **B325** (1989) 18.
- [3] W. Grimus and L. Lavoura, *One loop corrections to the seesaw mechanism in the multiHiggs doublet standard model*, *Phys.Lett.* **B546** (2002) 86–95, [[hep-ph/0207229](#)].
- [4] W. Grimus and L. Lavoura, *Soft lepton flavor violation in a multi Higgs doublet seesaw model*, *Phys.Rev.* **D66** (2002) 014016, [[hep-ph/0204070](#)].
- [5] D. Aristizabal Sierra and C. E. Yaguna, *On the importance of the 1-loop finite corrections to seesaw neutrino masses*, *JHEP* **1108** (2011) 013, [[arXiv:1106.3587](#)].
- [6] P. B. Dev and A. Pilaftsis, *Minimal Radiative Neutrino Mass Mechanism for Inverse Seesaw Models*, *Phys.Rev.* **D86** (2012) 113001, [[arXiv:1209.4051](#)].
- [7] D. Jurčiukonis, T. Gajdosik, A. Juodagalvis, and T. Sabonis, *Parametrizing the Neutrino sector of the seesaw extension in tau decays*, *PoS ICHEP2012* (2013) 372, [[arXiv:1212.5370](#)].
- [8] D. Jurciukonis, T. Gajdosik, A. Juodagalvis, and T. Sabonis, *Neutrino mass spectrum from the seesaw extension*, *Acta Phys.Polon.Supp.* **6** (2013) 675–680, [[arXiv:1212.6912](#)].
- [9] T. Gajdosik, A. Juodagalvis, D. Jurčiukonis, and T. Sabonis, *Progress in the parametrisation of the Neutrino sector*, *Acta Phys.Polon.* **B44** (2013), no. 11 2347–2352, [[arXiv:1310.2476](#)].
- [10] D. Jurciukonis, T. Gajdosik, and A. Juodagalvis, *Light neutrino mass spectrum with one or two right-handed singlet fermions added*, *Nucl. Part. Phys. Proc.* (2014) [[arXiv:1410.4443](#)].
- [11] T. Gajdosik, D. Juriukonis, and A. Juodagalvis, *Impact of Majorana Neutrinos to Hadronic Tau Decays*, *Nucl.Part.Phys.Proc.* **260** (2015) 257–259.
- [12] D. Forero, M. Tortola, and J. Valle, *Neutrino oscillations refitted*, *Phys.Rev.* **D90** (2014), no. 9 093006, [[arXiv:1405.7540](#)].
- [13] Z.-z. Xing, *A full parametrization of the 6×6 flavor mixing matrix in the presence of three light or heavy sterile neutrinos*, *Phys.Rev.* **D85** (2012) 013008, [[arXiv:1110.0083](#)].
- [14] H. E. Haber and D. O’Neil, *Basis-independent methods for the two-Higgs-doublet model III: The CP-conserving limit, custodial symmetry, and the oblique parameters S , T , U* , *Phys.Rev.* **D83** (2011) 055017, [[arXiv:1011.6188](#)].

- [15] **ATLAS, CMS** Collaboration, G. Aad et al., *Combined Measurement of the Higgs Boson Mass in pp Collisions at $\sqrt{s} = 7$ and 8 TeV with the ATLAS and CMS Experiments*, [arXiv:1503.07589](#).
- [16] H. E. Haber and D. O’Neil, *Basis-independent methods for the two-Higgs-doublet model. II. The Significance of $\tan\beta$* , *Phys.Rev.* **D74** (2006) 015018, [[hep-ph/0602242](#)].
- [17] P. B. Pal, *Dirac, Majorana and Weyl fermions*, *Am. J. Phys.* **79** (2011) 485–498, [[arXiv:1006.1718](#)].
- [18] M. Gell-Mann, P. Ramond, and R. Slansky, *Complex Spinors and Unified Theories*, in *Supergravity, Proceedings of the Workshop, Stony Brook, New York, Conf.Proc.* **C790927** (1979) 315–321, [[arXiv:1306.4669](#)].
- [19] J. Schechter and J. Valle, *Neutrino Masses in $SU(2) \times U(1)$ Theories*, *Phys.Rev.* **D22** (1980) 2227.
- [20] M. E. Peskin and T. Takeuchi, *Estimation of oblique electroweak corrections*, *Phys.Rev.* **D46** (1992) 381–409.
- [21] D. Eriksson, J. Rathsmann, and O. Stal, *2HDMC: Two-Higgs-Doublet Model Calculator Physics and Manual*, *Comput.Phys.Commun.* **181** (2010) 189–205, [[arXiv:0902.0851](#)].
- [22] G. Funk, D. O’Neil, and R. M. Winters, *What the Oblique Parameters S , T , and U and Their Extensions Reveal About the 2HDM: A Numerical Analysis*, *Int.J.Mod.Phys.* **A27** (2012) 1250021, [[arXiv:1110.3812](#)].
- [23] A. Ibarra and G. G. Ross, *Neutrino phenomenology: The Case of two right-handed neutrinos*, *Phys.Lett.* **B591** (2004) 285–296, [[hep-ph/0312138](#)].
- [24] W.-l. Guo, Z.-z. Xing, and S. Zhou, *Neutrino Masses, Lepton Flavor Mixing and Leptogenesis in the Minimal Seesaw Model*, *Int.J.Mod.Phys.* **E16** (2007) 1–50, [[hep-ph/0612033](#)].
- [25] G. C. Branco, M. Rebelo, and J. Silva-Marcos, *Leptogenesis, Yukawa textures and weak basis invariants*, *Phys.Lett.* **B633** (2006) 345–354, [[hep-ph/0510412](#)].
- [26] W. Rodejohann, M. Tanimoto, and A. Watanabe, *Relating large U_{e3} to the ratio of neutrino mass-squared differences*, *Phys.Lett.* **B710** (2012) 636–640, [[arXiv:1201.4936](#)].
- [27] L. Lavoura, W. Rodejohann, and A. Watanabe, *Reproducing lepton mixing in a texture zero model*, *Phys.Lett.* **B726** (2013) 352–355, [[arXiv:1307.6421](#)].
- [28] S. F. King, *Minimal predictive see-saw model with normal neutrino mass hierarchy*, *JHEP* **1307** (2013) 137, [[arXiv:1304.6264](#)].
- [29] S. Raby, *A Natural framework for bilarge neutrino mixing*, *Phys.Lett.* **B561** (2003) 119–124, [[hep-ph/0302027](#)].
- [30] R. Kuchimanchi and R. Mohapatra, *Bimaximal neutrino mixing from a local $SU(2)$ horizontal symmetry*, *Phys.Rev.* **D66** (2002) 051301, [[hep-ph/0207110](#)].
- [31] B. Dutta and R. Mohapatra, *Lepton flavor violation and neutrino mixings in a 3×2 seesaw model*, *Phys.Rev.* **D68** (2003) 056006, [[hep-ph/0305059](#)].
- [32] B. Dziewit, S. Zajac, and M. Zralek, *Majorana neutrino mass matrix with CP symmetry breaking*, *Acta Phys.Polon.* **B42** (2011) 2509–2516, [[arXiv:1204.3665](#)].